Bisecting families for set systems and related problems

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by

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Abstract

Given a family $\mathcal{A}$ of subsets of $[n]$, $n \in \mathbb{N}$, finding another family of optimal size satisfying a certain relationship with sets in $\mathcal{A}$ constitute a class of problems studied in extremal combinatorics. This class includes the set cover problem, the problem of separating families and its variants [59, 38, 71], and the test cover problem [50, 32, 21]. Another class of well studied problems is "Covering the $\{0,1\}^n$ Hamming cube with the minimum number of affine hyperplanes" - a point $x \in \{0,1\}^n$ is said to be *covered* by a hyperplane $H(a,b)$ if $\langle a, x \rangle = b$ (see [2, 43, 66, 65]). Viewing the elements of a family $\mathcal{A}$ as the points of the $\{0,1\}^n$ Hamming cube and putting restrictions on the covering hyperplanes $H(a,b)$, the covering hyperplanes correspond to a family $\mathcal{B}$ with interesting combinatorial connections with $\mathcal{A}$. In this thesis, we study few such connections, their variants and the underlying minimization problems.

Let $\mathcal{A}$ be a family of subsets of $[n]$, where $[n] = \{1, \ldots, n\}$. For any set $A \subseteq [n]$, let $\overline{A}$ denote the complement set of A, i.e. $\overline{A} = [n] \setminus A$. Given a $D \subseteq \{-n, -n+1, \ldots, 0, \ldots, n\}$, we say a family $\mathcal{B}$ is *D-secting* for $\mathcal{A}$ if for each subset $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ such that $|A \cap B| - |A \cap \overline{B}| = i$, where $i \in D$. A *D-secting* family $\mathcal{B}$ of $\mathcal{A}$, where $D = \{-1, 0, 1\}$, is a *bisecting* family ensuring the existence of a subset $B \in \mathcal{B}$ such that $|A \cap B| \in \{\lceil \frac{|A|}{2} \rceil, \lfloor \frac{|A|}{2} \rfloor\}$, for each $A \in \mathcal{A}$. Let $\beta_D(\mathcal{A})$ denote the minimum cardinality of a D-secting family for $\mathcal{A}$. Observe that if $D = \{i\}$, only those sets $A \in \mathcal{A}$ for which $|A| \equiv i \pmod 2$ and $|A| \geq i$ can attain a value of i for $|A \cap B| - |A \cap \overline{B}|$. So, we consider only those sets A in the family $\mathcal{A}$ for which $|A| \equiv i \pmod 2$ and $|A| \geq i$, when $D = \{i\}$. We define $\beta_D(n)$ as

$$\beta_D(n) = \max_{\mathcal{A}} \beta_D(\mathcal{A}).$$

Let Y denote a ± 1 bicoloring of elements of $[n]$, i.e. $Y : [n] \to \{+1, -1\}$. We abuse the notation to denote the subset of $[n]$ colored with +1 (-1) with respect to bicoloring Y

as $Y(+1)$ (respectively, $Y(-1)$). Note that to describe a bicoloring of $[n]$, it suffices to specify either $Y(+1)$ or $Y(-1)$. Allowing $B = Y(+1)$, for any $A \subseteq [n]$, $|A \cap B| - |A \cap \overline{B}|$ is equivalent to $|A \cap Y(+1)| - |A \cap Y(-1)|$ (this is called as *discrepancy* of A with respect to bicoloring Y). Therefore, $|A \cap B| - |A \cap \overline{B}|$ can represent the difference of +1 colored points and -1 colored points in any A with respect to a bicoloring Y, where $Y(+1) = B$. This connection to discrepancy also leads to the following reformulation in terms of covering the $\{0, 1\}^n$ Hamming cube.

For any subset $A \subseteq [n]$, let (i) $X_A = (x_1, \ldots, x_n) \in \{0, 1\}^n$ be the incidence vector such that $x_i = 1$ if and only if $i \in A$; and, (ii)$Y_A = (y_1, \ldots, y_n) \in \{-1, 1\}^n$ be the incidence vector such that $y_i = 1$ if and only if $i \in A$. Observe that for any two subsets A and B of $[n]$, the dot product of $X_A = (x_1, \ldots, x_n)$ with $Y_B = (y_1, \ldots, y_n)$, denoted by $\langle X_A, Y_B \rangle$, is equivalent to $|A \cap B| - |A \cap \overline{B}|$. Therefore, $\beta_D(n)$ may be alternatively defined as the minimum cardinality of a family $\mathcal{H}$ of structures $H(a, D)$ such that

1. $a \in \{-1, +1\}^n$ for each $H(a, D)$;

2. each $H(a, D)$ is a collection of $|D|$ hyperplanes $H(a, i)$, $i \in D$;

3. a point $x \in \{0, 1\}^n$ is said to be *covered* by some $H(a, D)$ if $\langle a, x \rangle \in D$;

4. for each $x \in \{0, 1\}^n$, there exists some $H(a, D) \in \mathcal{H}$ that covers x.

Varying the domain of a, D, and x leads to various kinds of combinatorial question (see Table 1). When a is restricted to the domain of $\{-1, 0, +1\}^n$ with exactly d nonzero coordinates (combinatorially, this can be viewed as a partial bicoloring of d out of n points), and $D = \{0\}$, the covering problem of the $\{0, 1\}^n$ Hamming cube translates into a special kind of D-secting family problem - "the Induced bisection problem". When the $\{0, 1\}^n$ Hamming cube is replaced with the $\{-1, +1\}^n$ Hamming cube, $a \in \{0, 1\}^n$, and $D = \{0\}$, the covering problem reduces to an inverse-D-secting family problem - " the System of unbiased representatives (SUR) problem". In the thesis, we study each of these notions in detail and establish bounds on cardinalities of such families.

x	a	$D \subseteq \{-n, \ldots, n\}$	combinatorial property
$\{0,1\}^n$	$\{-1,1\}^n$	D	D-section
$\{0,1\}^n$	$\{-1,0,1\}^n$ with d non-zeros	$\{0\}$	induced bisection
$\{-1,1\}^n$	$\{0,1\}^n$	$\{0\}$	SUR

Table 1: Various combinatorial notions corresponding to alterations in x, a and D.

Let $\mathcal{A}_e$ be a family of subsets of $[n]$ where each $A \in \mathcal{A}_e$ has an even cardinality. Recall that when D is restricted to the set $\{0\}$, any D-*secting* family $\mathcal{B}$ for $\mathcal{A}$ becomes a *bisecting* family for $\mathcal{A}_e$: for each subset $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ such that $|A \cap B| = \frac{|A|}{2}$. We have the following one family extension of the bisecting family notion. A family $\mathcal{F}$ of subsets of $[n]$ is called *bisection closed* if for each pair $A, B \in \mathcal{F}$, either A bisects B or B bisects A. We study extremal question regarding bisection closed families in detail and establish bounds on cardinalities of such families. We also study the problem of computation of bisecting families for products of families $\mathcal{A}$ for which $\beta_{[\pm 1]}(\mathcal{A})$ is known.

Keywords: discrepancy, hypergraphs, separating family, bisecting families, Hamming cube, covering, hyperplane, hypergraph bicoloring, hitting set, test cover, unbiased representatives

Chapter 1

Introduction

Extremal combinatorics is an important and rapidly developing area of Mathematics. It deals with problems of estimating the maximum (or minimum) size of a collection of some combinatorial objects such as sets, graphs or numbers, under certain restrictions. Some classical questions in extremal combinatorics include

- Túran's problem: What is the maximum number of edges in n vertex graph that does not contain a clique of size r, for some $2 \leq r \leq n$?
- Ramsey theoretic problem: What is the minimum n such that any bicoloring of edges of a K_n contains *large* monochromatic cliques?
- Erdós distinct distance problem: What is the minimal number of distinct distances between n points in a plane for any $n \in \mathbb{N}$?

Such problems often have direct applications to various fields of computer science including data structures, information theory, computer vision, robotics, bioinformatics, number theory, and cryptography. For instance, Chebyshev's proof of the 'Prime number theorem' on computing the number of primes less than n for any $n \in \mathbb{N}$ uses double counting in establishing both the lower bound as well the upper bound (see [67, chapter 5.1]). Moreover, deep connections between combinatorics and the 'mainstream' areas of mathematics like algebra, geometry, probability theory has been realized over the years. For instance, Brouwer's fixed-point theorem [15, 31] can be proved using the classical Sperner's lemma combined with an elegant pigeonhole argument (see [77]).

Given a family $\mathcal{F}$ of subsets of $[n]$, $n \in \mathbb{N}$, finding another family of optimal size satisfying certain relationships with sets in $\mathcal{F}$ constitutes a class of problems studied in extremal combinatorics. This class includes the set cover and related problems, the problems of separating families [59, 38, 71], and the test cover problem [50, 32, 21]. In the next section, we state the central problem addressed in this thesis.

1.1 D-secting families

Let $\mathcal{A}$ be a family of subsets of $[n]$, where $[n] = \{1, \ldots, n\}$. Another family $\mathcal{B}$ of subsets of $[n]$ is called a *bisecting family* for $\mathcal{A}$, if for each subset $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ such that $|A \cap B| \in \{\lceil \frac{|A|}{2} \rceil, \lfloor \frac{|A|}{2} \rfloor\}$. What is the minimum cardinality of a bisecting family for any family $\mathcal{A}$? For any set $A \subseteq [n]$, let $\overline{A}$ denote the complement set of A, i.e. $\overline{A} = [n] \setminus A$. We pose a more general problem based on the difference between $|A \cap B|$ and $|A \cap \overline{B}|$. We say a family $\mathcal{B}$ is D-*secting* for $\mathcal{A}$ if for each subset $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ such that $|A \cap B| - |A \cap \overline{B}| = i$, where $i \in D$, $D \subseteq \{-n, -n+1, \ldots, 0, \ldots, n\}$. Let $\beta_D(\mathcal{A})$ denote the minimum cardinality of a D-secting family for $\mathcal{A}$. Observe that if $D = \{i\}$, only those sets $A \in \mathcal{A}$ for which $|A| \equiv i \pmod 2$ and $|A| \geq i$ can attain a value of i for $|A \cap B| - |A \cap \overline{B}|$. So, we consider only those sets A in the family $\mathcal{A}$ for which $|A| \equiv i \pmod 2$ and $|A| \geq i$, when $D = \{i\}$. We define $\beta_D(n)$ as the maximum of $\beta_D(\mathcal{A})$ over all families $\mathcal{A}$ on $[n]$ and $\beta_D(n, k)$ as the maximum of $\beta_D(\mathcal{A})$ over all families $\mathcal{A} \subseteq \binom{[n]}{k}$. When $D = \{i\}$ ($D = \{-i, -i+1, \ldots, i\}$), we sometimes abuse the notation to denote $\beta_D(\mathcal{A})$ by $\beta_i(\mathcal{A})$ (respectively, $\beta_{[\pm i]}(\mathcal{A})$).

Significance of $|A \cap B| - |A \cap \overline{B}|$

Let Y denote a ± 1 bicoloring of elements of $[n]$, i.e. $Y : [n] \rightarrow \{+1, -1\}$. We abuse the notation to denote the subset of $[n]$ colored with +1 (-1) with respect to bicoloring Y as $Y(+1)$ (respectively, $Y(-1)$). Note that to describe a bicoloring of $[n]$, it suffices to specify either $Y(+1)$ or $Y(-1)$. Allowing $B = Y(+1)$, for any $A \subseteq [n]$, $|A \cap B| - |A \cap \overline{B}|$ is equivalent to $|A \cap Y(+1)| - |A \cap Y(-1)|$. Therefore, $|A \cap B| - |A \cap \overline{B}|$ represents the difference of the number of points colored +1 and -1 in any set A with respect to a bicoloring Y, where $Y(+1) = B$.

In various practical scenarios, it is required to minimize $\Big||A \cap B| - |A \cap \overline{B}|\Big|$. However, given a k-uniform family $\mathcal{A}$ of subsets of $[n]$, finding a single $B \subseteq [n]$, with $|A \cap B| - |A \cap \overline{B}| \in D$, $D = \{-k+1, \ldots, k-1\}$, for each $A \in \mathcal{A}$ directly reduces to a hypergraph bicolorability problem of k-uniform hypergraphs which is known to be NP-complete[45, 27]. Thus, finding a single $B \subseteq [n]$ even with almost all allowable values for D is computationally very hard. So, it is natural to allow multiple B's (or bicolorings) to ensure $|A \cap B| - |A \cap \overline{B}| \in D$ for some $B \in \mathcal{B}$, $D \subseteq \{-n+1, \ldots, n-1\}$. This collection of B's (bicolorings) naturally maps to a D-secting family $\mathcal{B}$ for $\mathcal{A}$. We analyze the interplay between the set of allowable values of $|A \cap B| - |A \cap \overline{B}|$ (i.e, D) and the cardinality of a D-secting family (i.e. $|\mathcal{B}|$). We provide a simple application of the study of D-secting families.

Example 1.1 (Network Management) *Consider a network of 10000 persons, and say, a million groups of persons. Each group is a subset of the set of persons. Each person can perform only one of two tasks in a day: either she can collect data or she can analyze data. For optimal operation in any group, it is advisable that the number of persons collecting data is equal to the number of persons analyzing data. However, we allow slight imbalances, whereby each group can operate provided the difference between the number of persons collecting data and the number of persons analyzing data is less than (say) 210, a tolerance limit. Now given this network, a scheduler has to assign tasks* *to persons satisfying the above constraints for a number of days, so that each group is deployed on at least one day. Note that the groups deployed for a day with tolerance 210 may be overlapping; in such cases, on that day, persons belonging to multiple groups will perform the same assigned task for the day in each group she belongs to. We wish to estimate the number of days required for such deployment of groups. It turns out that we can do this certainly within 10 days, irrespective of the composition of the million groups!*

Hamming cube and an alternate formulation

For any subset $A \subseteq [n]$, let (i) $X_A = (x_1, \dots, x_n) \in \{0,1\}^n$ be the incidence vector such that $x_i = 1$ if and only if $i \in A$; and, (ii)$Y_A = (y_1, \dots, y_n) \in \{-1,1\}^n$ be the incidence vector such that $y_i = 1$ if and only if $i \in A$. Observe that for any two subsets A and B of $[n]$, the dot product of $X_A = (x_1, \dots, x_n)$ with $Y_B = (y_1, \dots, y_n)$, denoted by $\langle X_A, Y_B \rangle$, is equivalent to $|A \cap B| - |A \cap \overline{B}|$. The *weight* of a vector $R = (x_1, \dots, x_n) \in \{0,1\}^n$ (or $\{-1,+1\}^n$) is the number of x_j's which are 1 (resp., -1), $1 \leq j \leq n$. Vector $R \in \{0,1\}^n$ is even (resp., odd) if the number of 1's in R is even (resp., odd). Given the n-dimensional Hamming cube $\{0,1\}^n$ and $D = \{-1,0,1\}$, $\beta_D(n)$ is the minimum cardinality of a set $\mathcal{V}$ of n-dimensional $\{-1,1\}$ vectors such that every point $X \in \{0,1\}^n$ of even weight of the Hamming cube has some $V \in \mathcal{V}$ which is orthogonal to X.

In Chapter 3, we consider this problem of Bisecting families and D-secting families along with its variation and restrictions in detail.

1.2 Bisection with restrictions

In the problem discussed in Example 1.1, suppose a further restriction is added: the scheduler cannot schedule more than 10 persons at a time and now he must ensure that for every group, there exists a day in which among the scheduled persons, there are exactly same number of persons collecting and analyzing data. What is the minimum number of days the scheduler must use for the worst case input network? This problem can be formalized as follows. Let G be a hypergraph on the vertex set $[n]$. Let Y^S denote a ± 1 bicoloring of vertices of $S \subseteq [n]$, i.e. $Y^S : S \to \{+1, -1\}$, for some $S \subseteq [n]$. We abuse the notation to denote the subset of vertices colored with +1 (-1) with respect to bicoloring Y^S as $Y^S(+1)$ (resp., $Y^S(-1)$). For a hyperedge $A \in E(G)$, let $A_{|S}$ denote $A \cap S$ - the hyperedge A induced on the subset $S \subseteq [n]$. A hyperedge $A \in E(G)$ is said to *induced bisected* by a bicoloring Y^S of a subset $S \subseteq V(G)$, if $|A_{|S}| \neq 0$ and $|A_{|S} \cap Y^S(+1)| = |A_{|S} \cap Y^S(-1)|$. It is not hard to see that the empty set and the singleton sets can never be induced bisected by a bicoloring. Additionally, the set $[n]$ cannot be induced bisected by a bicoloring consisting of exactly d colored points, when d is odd. We call the empty set and the singleton sets (and additionally, $[n]$ when d is odd) as *trivial*. A set $\mathcal{Y} = \{Y^{S_1}, \ldots, Y^{S_t}\}$ of t bicolorings is called an *induced bisecting family of order* d for a hypergraph G if

1. each $S_i \subseteq [n]$ has exactly d vertices, $1 \leq i \leq t$, and

2. for every non-trivial hyperedge $A \in E(G)$, $A_{|S_i}$ is induced bisected by Y^{S_i} for at least one i, $1 \leq i \leq t$.

Let $\beta^d(G)$ denote the minimum cardinality of an induced bisecting family of order d for hypergraph G.

Formulation based on the Hamming cube

Two n-dimensional vectors A and B, $A, B \in \mathbb{R}^n$, are said to be *trivially orthogonal* if in every coordinate $i \in [n]$, at least one of $A(i)$ or $B(i)$ is zero. The vectors A and B are *non-trivially orthogonal* if they are orthogonal, but not trivially orthogonal. For instance, the rows of a Hadamard matrix are non-trivially orthogonal. Consider the following problem: "Given the n-dimensional Hamming cube $\{0,1\}^n$, what is the minimum cardinality of a subset $\mathcal{V}$ of n-dimensional $\{-1, 0, 1\}$ vectors, each containing exactly d non-zero entries, such that every point $X \in \{0,1\}^n$ in the Hamming cube has some $V \in \mathcal{V}$ which is non-trivially orthogonal to X?". It is not hard to see that the all-zero vector and the unit vectors $\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,$ $(0,0,\ldots,1)\}$ can never have any non-trivially orthogonal vector in $\{-1,0,1\}^n$. Additionally, the all-ones vector $(1,\ldots,1)$ cannot be non-trivially orthogonal to any vector in $\{-1,0,1\}^n$ consisting of exactly d non-zero entries, when d is odd. We call the vectors $(0,\ldots,0),(1,0,\ldots,0),\ldots,(0,0,\ldots,1)$ (and additionally, $(1,\ldots,1)$ when d is odd) as *trivial*. Since no n-dimensional $\{-1,0,1\}$ vector with exactly one non-zero entry is non-trivially orthogonal to any non-trivial point of the Hamming cube, we assume that $d \geq 2$.

Definition 1.2 *Let $2 \leq d \leq n$, where d and n are integers. We define $\beta^d(n)$ as the minimum cardinality of a subset $\mathcal{V}$ of n-dimensional $\{-1,0,1\}$ vectors, each containing exactly d non-zero entries, such that every non-trivial point in the Hamming cube $\{0,1\}^n$ has a non-trivially orthogonal vector $V \in \mathcal{V}$.*

In Chapter 4, we consider this problem of induced bisecting families in detail and establish bounds for $\beta^d(n)$.

1.3 Some extremal questions related to bisection

Next, we consider the following extensions of the problems based on the property of bisection. A family $\mathcal{A}$ of subsets of $[n]$ is called *bisection closed* if for every $A, B \in \mathcal{A}$, either A bisects B or B bisects A. The problem is to estimate the largest cardinality of such a family. It is possible to compute the minimum bisecting families for some small families. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote two family of subsets of $[n]$, and let $\beta_D(\mathcal{A}_1)$ and $\beta_D(\mathcal{A}_2)$ are known for $D = \{-1,0,1\}$. We study the problem of computation of bisecting families for products of families $\mathcal{A}_1$ and $\mathcal{A}_2$.

Example 1.3 *Let $\mathcal{I}$ denote the family of subsets corresponding to the intervals on a line containing the points from 1 to n, i.e., $\mathcal{I} = \{I = (x, x+1, \ldots, y) | 1 \leq x < y \in$*

$[n]\}$. *It is not very hard to see that* $\beta_{[\pm 1]}(\mathcal{I})$ *is 1. Similarly, let* $\mathcal{I}^2$ *denote the family of subsets corresponding to the rectangular two dimensional intervals on a plane formed by Cartesian product* $[n] \times [n]$. *It is not immediate how to directly compute* $\beta_{[\pm 1]}(\mathcal{I}^2)$ *and it feels like we need at least two sets in any bisecting family to bisect all such two dimensional rectangular sets. However, using the fact that* $\mathcal{I}^2$ *is formed by Cartesian product* $\mathcal{I} \times \mathcal{I}$*, we can compute a bisecting family of cardinality one.*

1.4 The inverse problem of unbiased representation

In the questions related to bisecting families and D-secting families, we are given a family $\mathcal{A}$ of subsets of $[n]$ and the problem is to compute a set of bicolorings $\mathcal{B}$ such that for every $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ with $\langle X_A, Y_B \rangle \in D$. It is natural to ask the inverse problem: given a set of bicolorings $\mathcal{B}$, finding a small family of subsets satisfying the zero-dot-product property. This problem has numerous application in the field of drug testing and formation of unbiased committees as discussed in Chapter 5 in detail. Below, we define the problem formally.

Let $\mathcal{B}$ denote a set of bicolorings of $[n] = \{1, \ldots, n\}$, where each bicoloring $B \in \mathcal{B}$ maps each point in $[n]$ to either -1 or +1. Let Y_B denote the n-dimensional vector representing the bicoloring B, i.e. $Y_B = (B(1), \ldots, B(n))$. A non-empty set $A \subseteq [n]$ is said to be an *unbiased representative* for a bicoloring $B \in \mathcal{B}$ if $\langle X_A, Y_B \rangle = 0$, where X_A denotes the 0–1 n-dimensional incidence vector corresponding to A. We call a family $\mathcal{A}$ of subsets of $[n]$ an *unbiased representative family* (URF in short) for $\mathcal{B}$ if for every bicoloring $B \in \mathcal{B}$, there exists at least one set $A \in \mathcal{A}$ such that $\langle X_A, Y_B \rangle = 0$. Note that the two monochromatic bicolorings can never have any unbiased representatives - we call these bicolorings as 'trivial'. Let $\gamma(\mathcal{B})$ denote the minimum cardinality of an unbiased representative family for $\mathcal{B}$. We define the maximum of $\gamma(\mathcal{B})$ over all possible family $\mathcal{B}$ of non-trivial bicolorings as $\gamma(n)$.

In Chapter 5, we consider this problem of Unbiased representative families in detail, discuss its relation to bisecting families and establish bounds for $\gamma(n)$.

1.5 Organization of the thesis

In Chapter 2, we give a brief presentation of existing literature on the topics related to the problems addressed in the thesis, and discuss tools and techniques used in the thesis. In Chapter 3, we discuss the problem of bisecting and D-secting families, expand upon the ideas used and present the results obtained in this direction. In Chapter 4, we investigate a natural extension of the bisecting family problem- the induced bisecting family problem, study few extremal problems and obtain (in some cases) asymptotically tight bounds. In Chapter 5, we study the problem of 'systems of unbiased representatives' for a family of bicolorings- this can be viewed as a dual or inverse question to that posed for the bisecting family problem. In Chapter 6, we analyze few extremal questions pertaining to bisection closed families and product of set systems. In Chapter 7, we discuss some questions that remain open for investigation. From the introduction, it is clear that all of these problems can be either stated in a 'set-theoretic' framework or a 'hyperplane-covering' framework. We follow the set theoretic framework for the remainder of this report.

Chapter 2

Preliminaries and literature survey

Let n be some positive integer and $[n]$ denotes the set $\{1, \ldots, n\}$. Let $[\pm i]$ denote the set $\{-i, \ldots, 0, \ldots, +i\}$. A *set system* or *family of sets* denote a collection of subsets of $[n]$. Set systems are also referred to as *hypergraphs*. For any hypergraph G, $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set of G. A hypergraph is k-uniform if all its members are k-elements subsets of $[n]$. So, graphs are 2-uniform hypergraphs. This chapter is organized into two sections. In Section 1, we give a brief presentation of existing literature on topics related to the problems addressed in the thesis. In Section 2, we discuss the tools and techniques used in the thesis.

2.1 Existing problems and results

Discrepancy of set systems

A *vertex coloring* C of a hypergraph G is a function $C : V(G) \rightarrow \mathbb{N}$. An r-coloring of vertices is a function $C : V(G) \rightarrow \{1, ..., r\}$ i.e., an assignment of a color in range 1 through r to every vertex $v \in V$. A coloring C of vertices is *proper* if no $e \in E(G)$ remains monochromatic under C. The minimum number of colors required for a proper coloring of the vertices of G is the *chromatic* number $\chi(G)$ of G.

Definition 2.1 (Discrepancy) *Given a bicoloring X, $X : V(G) \rightarrow \{-1, +1\}$, let*

$\mathbb{C}_X(e) = |\sum_{v \in e} X(v)|$ *denote the discrepancy of the hyperedge* $e \in E(G)$ *under the bicoloring* X*. Then, the* combinatorial discrepancy *of the hypergraph* G*, denoted by* $disc(G)$*, is defined as* $disc(G) = \min_X \max_{e \in E} \mathbb{C}_X(e)$.

The earliest results in discrepancy theory come from ramsey theoretic problems such as the Van der Waerden Theorem [22] and Ramsey Theorem [58]; however, the first result that brought discrepancy theory proper attention was Roth's Theorem [62] on arithmetic progression. The first result establishing upper bounds on $disc(G)$ in terms of $|V(G)|$ was due to Olson and Spencer [54] who proved that $disc(G) \leq 2\sqrt{n \log n}$, where $n = |V(G)|$. Spencer [69] gave an alternate upper bound of $O(\sqrt{n \log(\frac{2m}{n})})$ using information theoretic arguments, this result is the *Spencer's theorem*. This result is particularly interesting due to the following corollary - 'when the hypergraph G has $O(n)$ hyperedges, $disc(G)$ is at most $6\sqrt{n}$'. Spencer's theorem has applications in Fourier analysis to "Rudin-Shapiro sequences" (see [69]) and in Littlewood's problem on "Flat polynomials" (see [11]). Spencer's method was non-constructive: recent works by Nikhil Bansal [8], Bansal and Spencer [9], Lovett and Meka [47] provide constructive approaches for obtaining low discrepancy colorings. Recently, combinatorial discrepancy and related notions has been applied to great effect in the study of optimal bin packing: given n items each having size in the range $[0, 1]$, packing the items in the smallest number of bins each of size 1. Though, the problem is known to be NP-hard [30, 18], an existing algorithm by Karmakar and Karp [37] obtains a packing using $OPT + O(\log^2 OPT)$ bins. Using the ideas of combinatorial discrepancy, Rothvoß [63], and Rothvoß and Hoberg [35] gave an algorithm that obtains a packing using $OPT + O(\log OPT \log\log OPT)$ and $OPT + O(\log OPT)$ bins, respectively. There is a huge literature of combinatorial discrepancy along with its measure theoretic counterparts; for definitions, results, and extensions of discrepancy and related problems, see [16, 48, 34, 12].

Discrepancy connection to our work

Below, we define $\beta_D(E(G))$ in terms of the discrepancy of a hypergraph G, where $D = [\pm i]$. Let $t \in \mathbb{N}$ be the minimum number such that there exists a set of t hypergraphs $G_1, \ldots, G_t$ on vertex set $V = [n]$ with (i) $disc(G_j) \in [\pm i]$, for $1 \leq j \leq t$, and, (ii) $\cup_{j=1}^{t} G_j = G$. Given an optimal D-secting family $\mathcal{B}$ of $E(G)$, it is easy to construct a set of hypergraphs $G_1, \ldots, G_{|\mathcal{B}|}$ satisfying the above conditions. Again, given a set of t hypergraphs $G_1, \ldots, G_t$ satisfying conditions (i) and (ii) under bicolorings $Y_1, \ldots, Y_t$, respectively, let $(Y_j(+1), Y_j(-1))$ be the bipartition of V formed by the bicoloring Y_j. Then, $\mathcal{B} = \{Y_1(+1), \ldots, Y_t(+1)\}$ is a D-secting family for $E(G)$. Thus, $\beta_{[\pm i]}(E(G)) = t$. Moreover, the discrepancy of a hypergraph G can be defined in terms of $\beta_{[\pm i]}(E(G))$ as follows. The *discrepancy* of a hypergraph G is the minimum $i \in \mathbb{N}$ such that $\beta_{[\pm i]}(E(G)) = 1$.

To see the connection of induced bisecting families with discrepancy, consider the following example. Let $t \in \mathbb{N}$ be the minimum number such that there exists a set of t hypergraphs $G_1, \ldots, G_t$ on subsets of $[n]$ with (i) $|V(G_j)| = d$, (ii) $disc(G_j) = 0$, for $1 \leq j \leq t$, and, (iii) for each $e \in E(G)$, there exists a G_j such that $e_{|V(G_j)} \in E(G_j)$ ($e_{|V(G)}$ denotes the hyperedge e induced on the vertex set $V(G)$). Then, $\beta^d(E(G)) = t$.

Covering the Hamming Cube

An affine hyperplane is a set of vectors $H(a, b) = \{x \in \mathbb{R}^n : \langle a, x \rangle = b\}$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. Covering the $\{0, 1\}^n$ Hamming cube with the minimum number of affine hyperplanes has been well studied - a point $x \in \{0, 1\}^n$ is said to be *covered* by a hyperplane $H(a, b)$ if $\langle a, x \rangle = b$. Without any further restriction, note that $H(e_1, 0)$ and $H(e_1, 1)$ covers every point on the $\{0, 1\}^n$ Hamming cube, where $e_1 = (1, 0, \ldots, 0)$ is the first unit vector. Alon and Füredi [2] show that the covering-by-hyperplanes problem becomes substantially nontrivial under the restriction that only the nonzero vectors are covered. They demonstrated, using the notion of Combinatorial Nullstellensatz [3], that we need at least n affine hyperplanes when the zero vector remains uncovered. This can be achieved by the set of hyperplanes $\{H(e_i, 1)\}$, where e_i is the ith unit vector, $1 \le i \le n$. Many other extensions of this covering problem involving other restrictions have been studied in detail (see [43, 66, 65]).

Relation to our work

The problem of bisecting families imposes the following constraints on the minimum cardinality set of covering hyperplanes $\{H_i(a_i, b_i)\}$: (i) $b_i \in \{-1, 0, 1\}$; (ii) $a_i \in \{-1, 1\}^n$. The problem of induced-bisecting families puts stronger restrictions not just on the hyperplanes, but also on the definition of 'covering' by a hyperplane $\{H_i(a_i, b_i)\}$: (i) $b_i = 0$; (ii) a_i consists of exactly d non-zero coordinates, $a_i \in \{-1, 0, 1\}^n$ and $d \in [n]$; (iii) we say a point x is *covered* by a hyperplane $H(a, b)$ when a is non-trivially orthogonal to x. The unbiased representative problem deals with covering of the $\{-1, 1\}^n$ Hamming cube with the following restrictions on the covering hyperplanes $\{H_i(a_i, b_i)\}$: (i) $b_i \in \{0\}$; (ii) $a_i \in \{0, 1\}^n$.

Separating families

Given a family $\mathcal{A}$ of subsets of $[n]$, finding another family $\mathcal{B}$ with certain properties has been well investigated. One of the most studied problem in this direction is the computation of *separating families*. Let $\mathcal{A}$ consist of pairs $\{i, j\}$, $i, j \in \mathbb{N}$, $i \neq j$ and $\mathcal{B}$ be another family of subsets on $[n]$ ($\mathcal{A}$ can be viewed as the edge set of a graph on vertex set $[n]$). A subset B separates a pair $\{i, j\}$ if $i \in B$ and $j \notin B$ or vice versa. The family $\mathcal{B}$ is a separating family for $\mathcal{A}$ if every pair $\{i, j\} \in \mathcal{A}$ is separated by some $B \in \mathcal{B}$. It is easy to see that $\mathcal{B}$ is indeed a bisecting family for $\mathcal{A}$. Let $f(n)$ denote the size of a minimum separating family $\mathcal{B}$ for a family $\mathcal{A}$ consisting of all the $\binom{n}{2}$ pairs (edge set of a complete graph on n vertices). Rényi [59] proved that $f(n) = \lceil \log n \rceil$. Observe that $f(n)$ is the minimum number of bipartite graphs needed to cover the edges of a complete graph K_n. We note the following generalization of the above statement for arbitrary graphs.

Proposition 2.2 (Folklore) *Let $\chi(G)$ denote the chromatic number of graph G. Then, $\lceil \log \chi(G) \rceil$ bipartite graphs are necessary and sufficient to cover the edges of G.*

Proof. Let $G_1(L_1 \cup R_1, E_1), \ldots, G_t(L_t \cup R_t, E_t)$ be t bipartite graphs whose union covers $G(V, E)$, $t \in \mathbb{N}$. Firstly, we show that $t \geq \lceil \log \chi(G) \rceil$. To each vertex $v \in V$, assign a t length 0,1 bit vector: jth bit is 1 if $v \in L_j$ and 0 otherwise. Color the vertices in V with the decimal equivalent of its bit vector. This uses at most 2^t colors and let this coloring be X. To see that X is a proper coloring of G, observe that for any edge $\{x, y\} \in E$ and $\{x, y\} \in E_j$, x and y receive different bits in jth position. So, they receive different colors under X. So, $2^t \geq \chi(G)$, i.e., $t \geq \lceil \log \chi(G) \rceil$.

To show that G can be covered with union of $\lceil \log \chi(G) \rceil$ bipartite graphs, consider a proper coloring $X : V(G) \to \{1, \ldots, \chi(G)\}$ using $\chi(G)$ colors. For each vertex $v \in V(G)$, obtain the $\lceil \log \chi(G) \rceil$ length 0,1 bit vector that is just the binary equivalent of its color under X. Construct graphs $G_1(L_1 \cup R_1, E_1), \ldots, G_{\lceil \log \chi(G) \rceil}(L_{\lceil \log \chi(G) \rceil} \cup R_{\lceil \log \chi(G) \rceil}, E_{\lceil \log \chi(G) \rceil})$ as follows: (i) add v to L_j if its jth bit is 1; otherwise add it to R_j, (ii) add edge $\{x, y\}$ to E_j if x and y have different bits in jth position. From construction, it is not hard to see that each $G_j(L_j \cup R_j, E_j)$, $1 \leq j \leq \lceil \log \chi(G) \rceil$, is bipartite. To see that $\cup_{j=1}^{\lceil \log \chi(G) \rceil} E_j = E$, for the sake of contradiction, assume that there exist an edge $\{x, y\} \in E$ such that $\{x, y\} \notin \cup_{j=1}^{\lceil \log \chi(G) \rceil} E_j$. Again from construction, $\{x, y\}$ must be monochromatic under X which is a contradiction. So, $\cup_{j=1}^{\lceil \log \chi(G) \rceil} E_j = E$ and we have shown that $\lceil \log \chi(G) \rceil$ bipartite graphs are sufficient to cover edges of G. This completes the proof of the proposition. $\square$

See [59, 38, 71, 24, 68] for detailed results and related problems on separating families.

Connection to bisecting families

Note that $f(n)$ is equal to $\beta_0(n, 2)$, thus $\beta_0(n, 2) = \lceil \log_2 n \rceil$. In fact, when the family $\mathcal{A}$ is the edge set of a graph G, where $V(G) = [n]$, any bisecting family $\mathcal{B}$ for $\mathcal{A}$ forms a covering of the edges of G with $|\mathcal{B}|$ bipartite graphs. We state these observations as a corollary below.

Corollary 2.2.1 *For a graph G, $\beta_0(E(G)) = \lceil \log_2 \chi(G) \rceil$. Thus, $\beta_0(n, 2) = \lceil \log_2 n \rceil$.*

Test Cover

Definition 2.3 *Given a family $\mathcal{A}$ of subsets of $[n]$, a sub-collection $\mathcal{T} \subseteq \mathcal{A}$ is a* test cover *for $[n]$ if every pair of $[n]$ is separated by some $S \in \mathcal{T}$: a subset S separates a pair $\{i, j\}$ if $i \in S$ and $j \notin S$ or vice versa.*

The test cover problem is studied in the context of drug testing, biology [57, 73, 42] and pattern recognition [23]. Garey and Johnson [30] first established that minimum test set problem is NP-hard by a reduction from the three dimensional matching problem. Moret and Shapiro [50] initiated the study of test cover problems in the following setting. A diagnostic table with n categories and m tests is an $n \times m$ matrix, where the (i, j)th entry denotes the result of the test T_j on an object from ith category. They showed that the set cover problem is polynomial time reducible to the test cover problem. Halldórson et al. [32] established that test cover is hard to approximate within $o(\log n)$ unless $P = NP$, where n is the number of elements in the ground set. They also give an $O(\ln k)$ factor approximation algorithm when the size of the largest set is upper bounded by k. Subsequently, the same problem and its extensions are studied in many other works (see [20, 19, 10]). Due to the intrinsic connection of the test cover problem with separating families, the connection with bisecting families follows.

In broad sense, our work is motivated by the ideas from *combinatorial discrepancy*, *separating families*, *test cover* and *covering the Hamming cube*. In the next section, we discuss tools and techniques from linear algebra, polynomials and probabilistic methods that are used in the thesis.

2.2 Tools and techniques

2.2.1 Linear algebraic methods

We denote an n-dimensional vector $X \in \{0,1\}^n$ (or $Y \in \{-1,+1\}^n$) as $X = (x_1, \ldots, x_n)$ (respectively, $Y = (y_1, \ldots, y_n)$) where $x_j \in \{0,1\}$ (resp., $y_j \in \{-1,+1\}$). The *weight* of a vector $X = (x_1, \ldots, x_n) \in \{0,1\}^n$ (or $\{-1,+1\}^n$) is the number of x_j's which are 1 (resp., -1), $1 \leq j \leq n$. A vector $X \in \{0,1\}^n$ is even (resp., odd) if the number of 1's in X is even (resp., odd). A vector $Y \in \{-1,1\}^n$ is even (resp., odd) if the number of -1's in Y is even (resp., odd). Let $\mathbb{F}$ denote a field, containing 0 and 1.

Definition 2.4 (Vector Space) *A vector space over a field $\mathbb{F}$ is an additive abelian group $(V, +, .)$ closed under left multiplication with elements of $\mathbb{F}$. It is required that multiplication is (i) distributive over addition in both V and $\mathbb{F}$, and (ii) associative in $\mathbb{F}$. Elements of V are called vectors or points.*

Given a set $\{v_1, \ldots, v_m\}$ of vectors, a *linear combination* of these vectors is of the form $\lambda_1 v_1 + \cdots + \lambda_m v_m$, where each $\lambda_i \in \mathbb{F}$. The set of all the linear combinations of $\{v_1, \ldots, v_m\}$ is the *span* of $\{v_1, \ldots, v_m\}$, denoted by $span\{v_1, \ldots, v_m\}$. The vectors $\{v_1, \ldots, v_m\}$ are *linearly independent* if $\lambda_1 v_1 + \cdots + \lambda_m v_m = 0$ implies that each $\lambda_i = 0$. A set of linearly independent vectors that spans the vector space V is called a *basis* for V. The cardinality of the basis set is the *dimension* of V, denoted as $dim(V)$.

Proposition 2.5 *Let $v_1, \ldots, v_k$ denote a collection of linearly independent vectors in a vector space V. Then, $k \leq dim(V)$.*

Proposition 2.5 is called as the *linear algebra* bound. Given two vectors $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ from some vector space V, their inner product, denoted by $\langle X, Y \rangle$, is $x_1 y_1 + \cdots + x_n y_n$. The vectors X and Y are *orthogonal* if $\langle X, Y \rangle$ is 0. For any subspace $U \subseteq V$, its orthogonal space denoted by $U^\perp$ is defined as

$$U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \text{ for each } u \in U\}.$$

The dimensions of these spaces are related by the following equality.

Proposition 2.6 *Let V denote a finite dimensional vector space and $U \subseteq V$ be a subspace of V. Then, $dim(U) + dim(U^{\perp}) = dim(V)$.*

Function space

Let S be any set and $\mathbb{F}$ denote some field. A function $f : S \to \mathbb{F}$ is a mapping of the elements of S to some elements of $\mathbb{F}$. Let V denote the set of all the functions from S to $\mathbb{F}$. For any two functions $f, g \in V$, let (i) $(f + g)(x) = f(x) + g(x)$ and (ii) $(cf)(x) = cf(x)$, where $x \in S$, $c \in \mathbb{F}$. Then, $(V, +, .)$ is a vector space over $\mathbb{F}$ called *function space*. The zero element is the function f such that $f(x) = 0$ for all $x \in S$. In the remaining of the discussion, we restrict our focus to the domain of $\{0, 1\}^n$ where $\mathbb{F} = \mathbb{R}$, the set of real numbers.

Proposition 2.7 (Diagonal criterion) *For $i \in \{1, \ldots, m\}$, let $f_i : S \to \mathbb{F}$ be functions and $v_j \in S$ such that*

$$f_i(v_j) \begin{cases} \neq 0, \textit{ for } i = j, \\ = 0, \textit{for } 1 \leq j < i \leq m. \end{cases}$$

Then, $f_1, \ldots, f_m$ are linearly independent.

Proof. For the sake of contradiction, assume that there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ such that at least one of them is nonzero and $\lambda_1 f_1 + \cdots + \lambda_m f_m = 0$. Let i be the smallest index such that λ_i is nonzero. Then, evaluating the functions on v_i, we get $\lambda_i = 0$, which is a contradiction. □

Polynomials

Let $\mathbf{R}$ denote a commutative ring with identity. Then, we can construct a ring $\mathbf{R}[x]$ of polynomials whose elements are of the form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$, where each $a_i \in \mathbf{R}$ and the symbol x is the indeterminate or variable. Each term is called a *monomial*. The degree of the polynomial is the maximum of the degrees of the monomials having nonzero coefficient. Note that the definition directly extends to polynomials on multiple indeterminates. In the rest of the discussion, we focus our discussion on polynomials in $\mathbb{F}(X = (x_1, \ldots, x_n))$ and the functions they compute on $\{0,1\}^n$ (or $\{-1,1\}^n$), where $\mathbb{F}$ denote a field containing both 0 and 1.

A polynomial is *multilinear* if each indeterminate in any monomial has degree at most one. In the domain of $\{0,1\}^n$ Hamming cube, any x_i^2 can be replaced by x_i. Similarly, in the domain of $\{-1,1\}^n$ Hamming cube, any x_i^2 can be replaced by 1. This process is often termed as *multilinearization*. Therefore, if the domain is restricted to either the $\{0,1\}^n$ Hamming cube or $\{-1,1\}^n$ Hamming cube, any polynomial of degree k can be replaced by a *multilinear* polynomial of degree at most k, where k is an integer. One can define any multilinear polynomial as polynomials of the form

$$P(X) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i.$$

Fact 1 *The space of multilinear polynomials over $X = (x_1, \ldots, x_n)$ form a vector space over $\mathbb{R}$ (the set of all real numbers), where the domain is restricted to the $\{0,1\}^n$ Hamming cube. The monomials of the form $\prod_{i \in S} x_i$ for all subsets $S \subseteq [n]$ forms a basis for the vector space.*

Let $f : \{0,1\}^n \to \{0,1\}$ denote a Boolean function on the $\{0,1\}^n$ Hamming cube. Based on the evaluations of f, there is a natural way to associate multilinear polynomials to Boolean functions.

Definition 2.8 *A multilinear polynomial $P(X)$* ***exactly represents*** *a Boolean function f if and only if for every $X \in \{0,1\}^n$, $P(X) = f(X)$.*

Proposition 2.9 *For every function $f : \{0,1\}^n \to \mathbb{R}$, there is an unique multilinear polynomial $P(X) \in \mathbb{R}[X]$ that exactly represents it.*

Proof. Let $I_A(X)$ be defined as follows.

$$I_A(X) = \left(\frac{1 + (1 - 2a_1)(1 - 2x_1)}{2}\right) \cdots \left(\frac{1 + (1 - 2a_n)(1 - 2x_n)}{2}\right).$$

Note that $I_A(X)$ is an indicator variable that is 1 if and only if $X = A$, where $A = (a_1, \ldots, a_n) \in \{0, 1\}^n$. Now, define $P(X)$ as

$$P(X) = \sum_{A \in \{0,1\}^n} f(A) I_A(X).$$

It is not hard to see that P exactly represents f.

To prove uniqueness, observe that if two distinct polynomials compute the same function, then their difference is a non-zero polynomial evaluating to 0 at every point in $\{0, 1\}^n$. However, it is not very hard to see that no non-zero polynomial can evaluate to zero at every point in $\{0, 1\}^n$. $\square$

We now move to another kind of polynomial representation of Boolean functions – functions of the form $f : B^k \to B$, where $B = \{0, 1\}$ (or $B = \{-1, +1\}$) and k is some non-negative integer. We restrict ourselves to the domain of $\{-1, +1\}^n$. Note that under this restriction, a polynomial representing the 'parity' function on $Y = (y_1 \ldots, y_n)$ is given by the monomial $y_1 \cdots y_n$. Let $sign : \mathbb{R} \setminus \{0\} \to \{-1, 1\}$ denote the function that maps all positive real numbers to 1 and all negative real numbers to -1.

Definition 2.10 (Weak representation) *A multilinear polynomial $P(Y = ((y_1 \ldots, y_n)))$ weakly represents a Boolean function f if and only if P is nonzero and for every $Y \in \{-1, 1\}^n$ where $P(Y)$ is nonzero, $sign(f(Y)) = sign(P(Y))$. The* ***weak degree*** *of a function f is the degree of lowest degree polynomial that weakly represents f.*

The next result was originally proved by [49] that deals with weak degree of parity functions.

Proposition 2.11 *The weak degree of the parity function on n variables is n.*

Proof. We only need to show that there is no polynomial of degree less than n that weakly represents the party function. Let ϑ denote the parity function on n variables. For the sake of contradiction, assume that $P(Y = (y_1, \ldots, y_n))$ is a polynomial of degree less than n and it weakly represents the parity function. Consider the vector space V of all multilinear polynomials over reals equipped with the following inner product:

$$\langle P, Q \rangle = \sum_{Y \in \{-1,+1\}^n} P(Y)Q(Y).$$

Since P weakly represents ϑ, from definition, it follows that $\langle \vartheta, P \rangle$ is nonzero. Since P is of degree less than n, it follows that for any monomial M of P, $\langle \vartheta, M \rangle$ is 0: M misses at least one variable in $y_1, \ldots, y_n$ (let that be y_j) and $\langle \vartheta, M \rangle = \sum_{Y \in \{-1,+1\}^n} P(Y)Q(Y) = \sum_{Y \in \{-1,+1\}^n, y_j = -1} P(Y)Q(Y) + \sum_{Y \in \{-1,+1\}^n, y_j = 1} P(Y)Q(Y) = 0$. This yields the desired contradiction. □

Nullstellensatz

The following version of Hilbert's Nullstellensatz and an extension also known as 'Combinatorial Nullstellensatz' is widely used in various combinatorial problems.

Theorem 2.12 (Hilbert's Nullstellensatz) *Let $\mathbb{F}$ be some field, $f \in \mathbb{F}(X = (x_1, \ldots, , x_n))$ be some polynomial, and $S_1, \ldots, S_n$ be nonempty subsets of $\mathbb{F}$. If $f(X) = 0$ for every $X \in S_1 \times \cdots \times S_n$, then there are polynomials $h_1, \ldots, h_n \in \mathbb{F}(X = (x_1, \ldots, , x_n))$ such that $deg(h_i) \leq deg(f) - |S_i|$ and*

$$f(X = (x_1, \ldots, , x_n)) = \sum_{i=1}^{n} h_i(X = (x_1, \ldots, , x_n)) \prod_{s \in S_i} (x_i - s).$$

Noga Alon [3] gave a simple proof of the above statement and then using it proved the following extension.

Theorem 2.13 (Combinatorial Nullstellensatz) *Let $\mathbb{F}$ be a field and $f \in \mathbb{F}(X = (x_1, \ldots, x_n))$ be some polynomial of degree d. Let the coefficients of the term $x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}$*

be non-zero and $t_1 + \ldots + t_n = d$. If $S_1, \ldots, S_n$ are finite sets with each $|S_i| \geq t_i + 1$, then there exists a point $X \in S_1 \times \cdots \times S_n$ such that $f(X)$ is nonzero.

Proof. Assume the result is false and $|S_i| = t_i + 1$. Let $g_i(X) = \prod_{s \in S_i}(x_i - s)$. Let $h_1, \ldots, h_n$ denote the polynomials guaranteed by Hilbert's Nullstellensatz. So,

$$deg(h_i) \leq deg(f) - |S_i| = d - (t_i + 1)$$

and $f(X) = \sum_{i=1}^{n} h_i(X)x_i^{t_i+1}$ + terms of degree less than $deg(f)$. By our assumption, the coefficient of the monomial $\prod_{i=1}^{n} x_i^{t_i}$ is nonzero in the left hand side - this implies that the coefficient of the monomial $\prod_{i=1}^{n} x_i^{t_i}$ is nonzero in the right hand side as well. However, degree of $h_i g_i$ is at most $deg(f)$ and if some monomial is of degree equal to $deg(f)$, it must be divisible by $x_i^{t_i+1}$. Therefore, it follows that coefficient of the monomial $\prod_{i=1}^{n} x_i^{t_i}$ is zero in the right hand side, which is a contradiction. $\square$

2.2.2 The probabilistic method

The probabilistic method is very useful in establishing bounds in various combinatorial problems. The main idea of the method is as follows. In order to show that a combinatorial object possesses a certain property, (i) one defines a suitable probability space, (ii) determines upper bounds on probabilities of certain 'bad' events that prevent the construction of the objects with the desired properties, and (iii) shows that with positive probability none of the 'bad' events occur. Let X denote a random variable taking values $\{x_1, \ldots, x_n\}$ where $X = x_i$ with probability $p(x_i)$. Then, the expected value $E[X]$ (or μ) is $\sum_{X=x} xp(X = x)$.

Fact 2 (Linearity of expectation) *If $X = c_1X_1 + \ldots + c_nX_n$ is the sum of n random variables $X_1, \ldots, X_n$, where $c_i \in \mathbb{R}$, then $E[X] = c_1E[X_1] + \ldots + c_nE[X_n]$.*

The property of linearity of expectation does not require any restriction on the random variables. In hindsight, it establishes that for any probability space, there exists a point for which $X \geq E[X]$ and a point for which $X \leq E[X]$. The following extension of these ideas gives the probability of the random variable exceeding certain values.

Concentration bounds

Theorem 2.14 (Markov's inequality) *For any non negative random variable X and $a > 0$, $P(X \geq a) \leq \frac{E[x]}{a}$.*

The variance $Var[X]$ (or σ^2) is $E[(X - E[X])^2]$. The following extension of Markov's inequality, also known as the *second moment method*, gives the probability of the difference of the random variable from its expectation exceeding certain values.

Theorem 2.15 (Chebyschev's inequality) *For any random variable X and $a > 0$, $P(|X - E[X]| \geq a) \leq \frac{\sigma^2}{a^2}$.*

An important difference from the Markov's inequality is that unlike Markov's inequality, Chebyschev's inequality works for random variables taking negative values as well. We use Chebyschev's inequality in Chapter 5 to obtain representative families with small bias. One of the applications of the second moment method is in establishing tight bounds for the covering number $Cov(\mathcal{F})$ for an r-uniform hypergraph $\mathcal{F}$ on $[n]$.

Definition 2.16 *A* covering *of a hypergraph on $[n]$ is a collection of hyperedges whose union covers the ground set $[n]$. The covering number $Cov(\mathcal{F})$ is the minimum cardinality of any covering of a hypergraph $\mathcal{F}$.*

Frankl and Ródl developed a method for obtaining such coverings provided the r-uniform hypergraph $\mathcal{F}$ on $[n]$ satisfies certain constraints.

Theorem 2.17 (Ródl nibble) *For every $r \geq 2$, real numbers $k \geq 1$, $a > 0$, there exists $\delta = \delta(r, k, a)$ and $d_0 = d_0(r, k, a)$ such that for every $n \geq D \geq d_0$, the following holds.*

Every r-uniform hypergraph $\mathcal{F}$ on $[n]$ with each vertex having a positive degree and satisfying the following conditions -

1. *for all but δn vertices $x \in [n]$, the degree $d(x) \leq (1 \pm \delta)D$,*

2. *for all vertices $x \in [n]$, $d(x) < kD$, and*

3. *for every pair of vertices $x, y \in [n]$, $x \neq y$, the codegree $d(x, y) \leq \delta D$,*

contains a cover of size at most $\frac{n}{r}(1 + a)$ hyperedges.

This is a very powerful method which has been useful in numerous applications in recent years. We use this theorem in Chapter 5 to obtain representative families consisting of fixed sized subsets.

In general, for independent and identically distributed Bernoulli random variables, there is much higher concentration around the mean than that given by Chebyschev's inequality. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables each taking values 1 and 0 with probabilities p and $1 - p$, respectively. Let $X = X_1 + \ldots + X_n$. Note that $E[X_i] = p$, so $E[X] = np$. We obtain stronger bounds for $P(X \geq (1+\delta)\mu)$ as follows. For a fixed $\lambda > 0$,

$$P(X \geq (1 + \delta)\mu) = P(e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}) \tag{2.1}$$

$$\leq \frac{E[e^{\lambda X}]}{e^{\lambda(1+\delta)\mu}} \qquad \text{(using Markov's inequality)} \tag{2.2}$$

$$E[e^{\lambda X}] = \prod_{i=1}^{n} E[e^{\lambda X_i}] = \prod_{i=1}^{n}(pe^{\lambda} + (1 - p)e^0) \leq \prod_{i=1}^{n} e^{p(e^{\lambda}-1)} = e^{np(e^{\lambda}-1)} = e^{\mu(e^{\lambda}-1)}.$$

So, from Inequality 2.1, we get, $P(X \geq (1 + \delta)\mu) \leq \frac{e^{\mu(e^{\lambda}-1)}}{e^{\lambda(1+\delta)\mu}}$. Substituting λ with $\ln(1 + \delta)$, we get the following theorem known as the *Chernoff's bound*.

Theorem 2.18 (Chernoff's bound)

$$P(X \geq (1 + \delta)\mu) \leq \left(\frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}}\right)^{\mu}.$$

$$P(X \leq (1 - \delta)\mu) \leq \left(\frac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}}\right)^{\mu}.$$

Lovász Local Lemma

A combinatorial structure may exclude or forbid several substructures. The presence of such forbidden structures may be viewed as failure events in probabilistic experiments that randomly generate structures. If a failure in realizing a combinatorial object is caused by the occurrence of any one or more of several mutually independent events, then it is easy to compute the probability that none of the independent failure-causing events occur. The Lovász Local Lemma addresses a generalized version of this problem where each of these failure events is independent of all but a few other failure-causing events. Such limited dependencies occur in several problems and the use of the local lemma is indeed beneficial. We first introduce the notion of a *dependency graph* as follows.

Definition 2.19 *The dependency graph for a set of events $E_1, E_2, \ldots, E_n$ is a directed graph G, where $V(G) = \{E_1, E_2, \ldots, E_n\}$, and event E_i is mutually dependent only on events in the set $\{E_j | (E_i, E_j) \in E(G)\}$.*

Suppose $E_1, E_2, \ldots, E_n$ is the set of bad events in some probability space Ω, each formed by some set of decision variables. We wish to know whether it is possible that in a random assignment to *decision variables*, none of the bad events E_i, $1 \leq i \leq n$, occur. Here, *decision variables* are assigned randomly and independently, and events are defined over these variables. It may however be the case that some bad events may occur in any random sample of the decision variables. Lovász introduced certain constraints under which it is possible to avoid all bad events simultaneously.

Lemma 2.20 (Lovász Local Lemma) *Let G be a dependency graph for events $E_1, \ldots, E_n$ in a probability space. Suppose that there exists $x_i \in [0, 1]$ for $1 \leq i \leq n$ such that $P(E_i) \leq x_i \prod_{\{i,j\} \in E} (1 - x_j)$, then $P\left(\bigcap_{i=1}^{n} \overline{E_i}\right) \geq \prod_{i=1}^{n} (1 - x_j)$.*

For the proof of the lemma, refer to [52]. Following corollary is a direct consequence of the lemma.

Corollary 2.20.1 *If every event $E \in \{E_1, \ldots, E_n\}$ is dependent on at most d other events, $P(E) \leq p$, and $ep(d+1) \leq 1$, then $P\left(\bigcap_{i=1}^{n} \overline{E_i}\right) > 0$.*

2.2.3 The entropy method

Information theoretic arguments are the basis of many results in combinatorics. The ideas in this field come from the encoding of one group of combinatorial objects in form of another group of combinatorial object whose properties are known. This can be best explained with the following example. Let A denote a set of n elements, where each element of A must be encoded as a unique binary string. What is the minimum length t such that such an encoding is possible? To answer this question, we only need to argue that - given any t, there are 2^t distinct t bit binary strings and 2^t must be at least n. The entropy method provides a general framework for such kind of problems.

Definition 2.21 *Let X denote a random variable taking values in some set S. The entropy $H(X)$ of X over a range S is*

$$H(X) = -\sum_{x \in S} P(X = x) \log_2 P(X = x).$$

If Y is another random variable taking values from T and (X, Y) is the random variable taking values from $S \times T$ based on the joint distribution of X and Y, then the conditional entropy $H(X|Y)$ is $H(X|Y) = H(X, Y) - H(Y)$.

Entropy is a measure of randomness of a variable. So once a variable is completely determined, its entropy is 0.

Fact 3 *Let X, Y and Z denote three random variables taking values from S, T, and R, respectively. Then,*

1. $H(X) \leq \log |S|$.

2. $H(X) \leq H(X, Y) \leq H(X) + H(Y)$.

3. $H(X|Y,Z) \leq \log|S|$.

We state a few applications of the entropy method to covering problems yielding (almost) tight lower bounds.

Covering a K_n with bipartite graphs

Proposition 2.22 *Let $G_1,\ldots,G_t$ denote bipartite graphs on $[n]$ as the vertex set such that $\cup_{i=1}^t G_i = K_n$. Let $size(G_i)$ denote the number of non-isolated vertices in G_i. Then, $t \geq \log_2(n)$. Moreover, $\sum_{i=1}^t size(G_i) \geq n\log n$.*

Proof. Let $G_i(A_i, B_i, E_i)$ denote the ith bipartite graph, with $A_i \cap B_i = \phi$ and $A_i \cup B_i \subseteq [n]$. Pick a vertex v randomly and uniformly. It follows from the definition of entropy that $H(v) = \log n$. Let χ_i be the indicator variable,

$$\chi_i = \begin{cases} 1, & \text{if } v \in A_i, \\ 0, & \text{otherwise.} \end{cases}$$

Note that once $\chi_1,\ldots,\chi_t$ are known, v is completely determined. So, $H(v|(\chi_1,\ldots,\chi_t)) = 0$. So, $0 = H(v|(\chi_1,\ldots,\chi_t)) = H(v,(\chi_1,\ldots,\chi_t)) - H((\chi_1,\ldots,\chi_t)) \geq H(v) - \sum_{i=1}^t H(\chi_i) \geq \log n - t$. This completes the proof of the first part.

To prove the second part, for every G_i, delete one of A_i or B_i with probability $\frac{1}{2}$ and remove corresponding vetices from the K_n. Let m_v denote the number of G_i's vertex v appears as non-isolated. Then, probability that v survives is 2^{-m_v}. So, expected number of vertices surviving is $\sum_{v\in[n]} 2^{-m_v}$. Since at most one vertex can survive at the end, by linearity of expectation, $\sum_{v\in[n]} 2^{-m_v} \leq 1$. Using the fact that 'arithmetic mean is at least the geometric mean' and $\sum_v m_v = \sum_{i=1}^t size(G_i)$, it follows that $\sum_{i=1}^t size(G_i) \geq n\log n$. $\square$

Proposition 2.23 (Fredman and Komlós) *Let $H_1,\ldots,H_t$ denote r-uniform r-partite hypergraphs on $[n]$ as the vertex set such that $\cup_{i=1}^t H_i$ is K_n^r- the complete r-uniform* *hypergraph on n vertices. Then,*

$$t \geq \frac{\binom{n}{r-2}(n-r+2)\log(n-r+2)}{2(\frac{n}{r})^{r-1}\binom{r}{2}} = \Omega(\frac{e^r}{r\sqrt{2\pi r}}\log n).$$

Proof. With every r-uniform hypergraph, associate a graph $G(H)$ as follows.

$$V(G(H)) = \{(S,i) | S \in \binom{[n]}{r-2}, i \in [n] \setminus S\}.$$

$$E(G(H)) = \{\{(S,i),(S,j)\} : S \cup \{i,j\} \in E(H)\}.$$

It follows from the premise of the theorem that $\cup_{i=1}^{t} G(H_i) = G(K_n^r)$. $G(K_n^r)$ can be viewed as union of $\binom{n}{r-2}$ complete K_{n-r+2} graphs, one corresponding to each $S \in \binom{[n]}{r-2}$, which are disjoint from one another. So, from Proposition 2.22, $\sum_{i=1}^{t} size(G(H_i))$ must be at least $\binom{n}{r-2}(n-r+2)\log(n-r+2)$. Since each H_i is r-partite, for any fixed $(r-2)$-sized part P, there are $(\frac{n}{r})^{r-2}$ distinct ways to choose one element each from each part of P and each one of them contribute at most $2(\frac{n}{r})$ non-isolated vertices to $size(G(H_i))$. Therefore, using Proposition 2.22,

$$t \cdot \binom{r}{r-2}\left(\frac{n}{r}\right)^{r-2} 2(\frac{n}{r}) \geq \sum_{i=1}^{t} size(G(H_i)) \geq \binom{n}{r-2}(n-r+2)\log(n-r+2).$$

□

Chapter 3

Bisecting and D-secting families for set systems

3.1 Introduction

Let n be any positive integer and $\mathcal{A}$ be a family of subsets of $[n]$. Another family $\mathcal{B}$ of subsets of $[n]$ is called a *bisecting family* for $\mathcal{A}$, if for each subset $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ such that $|A \cap B| \in \{\lceil \frac{|A|}{2} \rceil, \lfloor \frac{|A|}{2} \rfloor\}$. What is the minimum cardinality of a bisecting family for any family $\mathcal{A}$? We pose a more general problem based on the difference between $|A \cap B|$ and $|A \cap \overline{B}|$, where $\overline{B} = [n] \setminus B$. We say a family $\mathcal{B}$ is D*-secting* for $\mathcal{A}$ if for each subset $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ such that $|A \cap B| - |A \cap \overline{B}| = i$, where $i \in D$, $D \subseteq \{-n, -n+1, \ldots, 0, \ldots, n\}$. Let $\beta_D(\mathcal{A})$ denote the minimum cardinality of a D-secting family for $\mathcal{A}$. In particular, when $D = \{-1, 0, 1\}$, the family $\mathcal{B}$ becomes a bisecting family for $\mathcal{A}$. We study two cases depending on D: (i) $D = \{-i, -i+1, \ldots, 0, \ldots, i\}$, and (ii) $D = \{i\}$, for some $i \in [n]$. Observe that if $D = \{i\}$, only those sets $A \in \mathcal{A}$ for which $|A| \equiv i \pmod 2$ and $|A| \geq i$ can attain a value of i for $|A \cap B| - |A \cap \overline{B}|$. So, we consider only those sets A in the family $\mathcal{A}$ for which $|A| \equiv i \pmod 2$ and $|A| \geq i$, when $D = \{i\}$. We define $\beta_D(n)$ as the maximum of $\beta_D(\mathcal{A})$ over all families $\mathcal{A}$ on $[n]$ and $\beta_D(n, k)$ as the maximum of

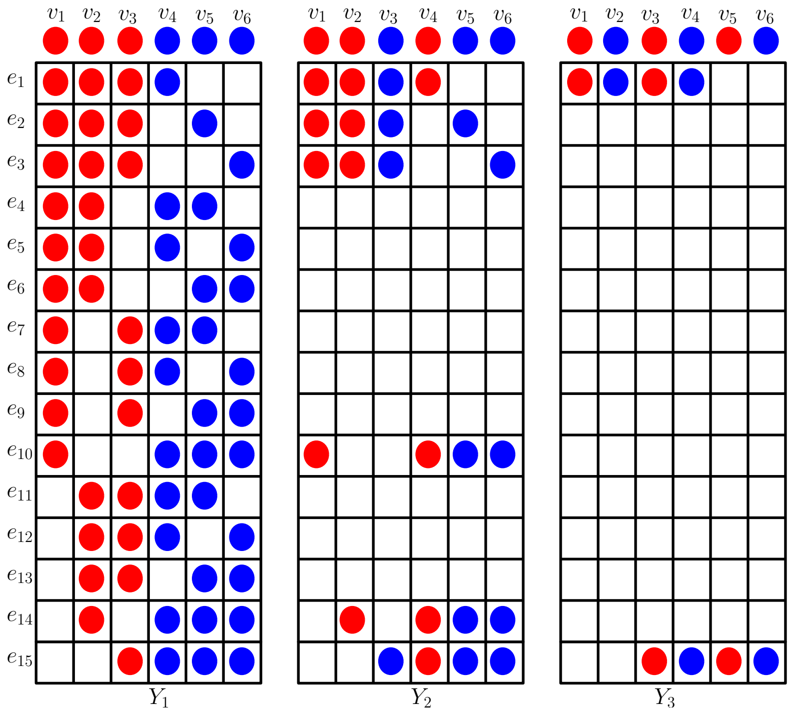

Figure 3.1: $Y_1 = \{\{v_1, v_2, v_3\} \to -1, \{v_4, v_5, v_6\} \to 1\}$, $Y_2 = \{\{v_1, v_2, v_4\} \to -1, \{v_3, v_5, v_6\} \to 1\}$, $Y_3 = \{\{v_1, v_3, v_5\} \to -1, \{v_2, v_4, v_6\} \to 1\}$ are bicolorings of $\{v_1, \ldots, v_6\}$. The sets $e_4, \ldots, e_9, e_{11}, e_{12}, e_{13}$ have same number of -1's and +1's in Y_1. Out of the remaining sets in $\binom{[6]}{4}$, e_2, e_3, e_{10}, e_{14} have same number of -1's and +1's in Y_2. Remaining sets e_1 and e_{15} have same number of -1's and +1's in Y_3. $\mathcal{B} = \{Y_1(-1), Y_2(-1), Y_3(-1)\} = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_5\}\}$ forms a bisecting family for $\binom{[6]}{4}$.

$\beta_D(\mathcal{A})$ over all families $\mathcal{A} \subseteq \binom{[n]}{k}$. When $D = \{i\}$ ($D = \{-i, -i+1, \ldots, i\}$), we sometimes abuse the notation to denote $\beta_D(\mathcal{A})$ by $\beta_i(\mathcal{A})$ (respectively, $\beta_{[\pm i]}(\mathcal{A})$).

Consider an example family $\mathcal{A}$ which consists of all the 4-element subsets of $\{1, \ldots, 6\}$. Note that since each subset $A \in \mathcal{A}$ has an even cardinality, $\beta_0(\mathcal{A}) = \beta_{[\pm 1]}(\mathcal{A})$. Let $\mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}\}$. It is not hard to verify that every 4-element subset $A \in \mathcal{A}$ is bisected by at least one element in $\mathcal{B}$ (see Figure 3.1). So, $\beta_0(\mathcal{A}) \leq 3$, for $\mathcal{A} = \binom{[6]}{4}$. In fact there is no pair of subsets of $\{1, \ldots, 6\}$ such that every 4-element subset $A \in \mathcal{A}$ is bisected by one of them, which is asserted by Proposition 3.16. Therefore, $\beta_0(\mathcal{A}) = 3$.

3.1.1 Notations and definitions

Let $[n]$ denote the set of integers $\{1,\ldots,n\}$, $\pm i$ denote the set of integers $\{-i,i\}$, and $[\pm i]$ denote the set of integers $\{-i,-i+1,\ldots,i\}$. Let $\mathcal{A}$ denote a family of subsets of $[n]$ and $\mathcal{B}$ denote another family of subsets with some desired intersection property with elements of $\mathcal{A}$. Let $\binom{[n]}{k}$ denote the family of all the k-sized subsets of $[n]$. We use $\beta_{[\pm i]}(\mathcal{A})$ (resp., $\beta_i(\mathcal{A})$) to denote $\beta_D(\mathcal{A})$ if $D=[\pm i]$ (resp., $D=\{i\}$). We denote an n-dimensional vector $X\in\{0,1\}^n$ (or $\{-1,+1\}^n$) as $X=(x_1,\ldots,x_n)$ where $x_j\in\{0,1\}$ (resp., $\{-1,+1\}$). The *weight* of a vector $X=(x_1,\ldots,x_n)\in\{0,1\}^n$ (or $\{-1,+1\}^n$) is the number of x_j's which are 1 (resp., -1), $1\leq j\leq n$. Vector $X\in\{0,1\}^n$ is even (resp., odd) if the number of 1's in X is even (resp., odd). A vector $Y\in\{-1,1\}^n$ is even (resp., odd) if the number of -1's in Y is even (resp., odd). We use $\log$ to denote $\log_2$ unless specified explicitly.

3.1.2 Chapter outline

We begin by addressing the problem of bounding and computing $\beta_D(n)$, where $D=[\pm i]$. We demonstrate a construction yielding an upper bound of $\lceil\frac{n}{2i}\rceil$ for $\beta_{[\pm i]}(n)$. Further, we show using a polynomial representation for the parity function that $\lceil\frac{n}{2i}\rceil$ is also a lower bound for $\beta_{[\pm i]}(n)$.

Theorem 3.1 $\beta_{[\pm i]}(n)=\lceil\frac{n}{2i}\rceil$, $n\in\mathbb{N}$, $i\in[n]$.

We study $\beta_{[\pm i]}(\mathcal{A})$ for a family $\mathcal{A}$ on $[n]$, in terms of i and $|\mathcal{A}|$, using Chernoff's bound.

Theorem 3.2 *Let $\mathcal{A}$ be a family of subsets of $[n]$ and let $m=|\mathcal{A}|$. Let $D=[\pm i]$, where $i\geq\sqrt{\frac{3n\ln(2m)}{t}}$ and $t\leq\frac{1}{2}\log m$. Then, $\beta_D(\mathcal{A})\leq t$.*

In particular, if $i\geq\sqrt{4.2n+1}$ and $|\mathcal{A}|=O(n^c)$, for $c\in\mathbb{N}$, a D-secting family $\mathcal{B}$ of cardinality $O(\log n)$ can be computed for families $\mathcal{A}$, thus improving the bound from Theorem 3.1 for this range of i and $|\mathcal{A}|$.

Subsequently, we study $\beta_D(n)$, where D is a singleton set, i.e., $D = \{i\}$. Note that $\beta_i(n) = \beta_{-i}(n)$. Moreover, when $D = \{-i, i\}$, note that $\beta_{\pm i}(n) \leq \beta_i(n) \leq 2\beta_{\pm i}(n)$. Therefore, we focus on establishing bounds for $\beta_i(n)$. We demonstrate a construction to show that $\beta_1(n)$ is at most $\lceil \frac{n}{2} \rceil$. We also show that $\beta_1(n)$ is at least $\lceil \frac{n}{2} \rceil$ using arguments similar to those in the proof of Theorem 3.1 about $\beta_{[\pm 1]}(n)$. In Section 3.3.2, we establish a lower bound of $\frac{n-i+1}{2}$ for arbitrary $i \in [n]$, $i \geq 2$. We demonstrate a construction establishing $\beta_i(n) \leq n - i + 1$. We have the following theorem.

Theorem 3.3 $\frac{n-i+1}{2} \leq \beta_i(n) \leq n - i + 1$, $n \in \mathbb{N}$, $i \in [n]$.

In Section 3.4, we consider families $\mathcal{A}$, $\mathcal{A} \subseteq \binom{[n]}{k}$. We study $\beta_{[\pm 1]}(n, k)$ in detail when k is even; the analysis for $\beta_i(n, k)$ for $i \in [n]$ and for the case when k is odd is analogous. We have lower bounds for $\beta_{[\pm 1]}(n, k)$ given by Theorem 3.4, Observation 3.9 (see Section 3.1.3), and Theorem 3.5 which are useful when k is a constant, k is sublinear in n, and k is linear in n, respectively. We establish the following theorem using entropy based arguments.

Theorem 3.4

$$\beta_{[\pm 1]}(n, k) \geq \begin{cases} \log(n - k + 2), \textit{ when } k \textit{ is even and } \frac{k}{2} \textit{ is odd,} \\ \lceil (\log \lceil \frac{n}{\lceil \frac{k}{2} \rceil} \rceil) \rceil, \textit{ for any } k \geq 2. \end{cases}$$

When $cn < k < (1 - c)n$ for a constant c, $0 < c < \frac{1}{2}$, we establish an improved lower bound for $\beta_{[\pm 1]}(n, k)$ using a vector space orthogonality argument, enabling us to apply a recent result of Keevash and Long [39].

Theorem 3.5 *Let c be a constant such that $0 < c < \frac{1}{2}$ and $n \in \mathbb{N}$. If $cn < k < (1-c)n$, then*

$$\max \Big\{ \beta_{[\pm 1]}(n, k), \beta_{[\pm 1]}(n, k-1), \beta_{[\pm 1]}(n, k-2), \beta_{[\pm 1]}(n, k-3) \Big\} \geq \delta n,$$

where $\delta = \delta(c)$ is some real positive constant.

Let $\mathcal{A}$ be a family of subsets of $[n]$. The *dependency* of a subset $A \in \mathcal{A}$ denoted by $d(A, \mathcal{A})$ is the number of subsets $\widehat{A} \in \mathcal{A}$, such that (i) $|A \cap \widehat{A}| \geq 1$, and (ii) $A \neq \widehat{A}$. The *dependency of a family* $d(\mathcal{A})$ or simply d, denotes the maximum dependency of any subset A in the family $\mathcal{A}$. We study $\beta_{[\pm 1]}(\mathcal{A})$ for families $\mathcal{A}$ consisting of k-sized sets with bounded dependency and using a corollary of the Lovász local lemma from [51], we prove the following probabilistic upper bound.

Theorem 3.6 *For a family $\mathcal{A}$ consisting of k-sized subsets of $[n]$ and dependency d, $\beta_{[\pm 1]}(\mathcal{A}) \leq \frac{\sqrt{k}}{c}(\ln(d+1)+1)$, where $c = 0.67$.*

We also study the case when $\mathcal{A}$ consists of all the subsets of $[n]$ of cardinality more than k, $k \in [n]$ and we have the following bounds.

Theorem 3.7 *Let $\mathcal{A} = \binom{[n]}{k} \cup \binom{[n]}{k+1} \ldots \cup \binom{[n]}{n}$. Then, $\frac{n-k+1}{2} \leq \beta_{[\pm 1]}(\mathcal{A}) \leq \min\{\frac{n}{2}, n - k + 1\}$.*

Note that when $n - k$ is a constant, Theorem 3.7 gives better upper bounds for $\beta_{[\pm 1]}(\mathcal{A})$.

3.1.3 Some quick observations

In this section, we derive a few basic results on $\beta_D(\mathcal{A})$, $\beta_D(n)$ and $\beta_D(n,k)$. $\mathcal{P}$ is a *property* for a set system if it is invariant under *isomorphism*. Two set systems $H = (X; E_1, E_2, \ldots, E_m)$ and $I = (Y; F_1, F_2, \ldots, F_m)$ are said to be isomorphic if they have the same number m of subsets, and if there exists a bijection $\varphi : X \to Y$ and a permutation π on $M = \{1, 2, \ldots, m\}$ such that

$$\varphi(E_i) = F_{\pi(i)} \ \ (i = 1, 2, \ldots, m).$$

See [13, page 411] for related notions. It is not hard to see that for any two isomorphic families $\mathcal{A}_1$ and $\mathcal{A}_2$ on $[n]$, $\beta_D(\mathcal{A}_1) = \beta_D(\mathcal{A}_2)$. So, $\beta_D(\mathcal{A}) \leq t$, for an integer t is a property of the set system. For any two families $\mathcal{A}_1$ and $\mathcal{A}_2$, $\mathcal{A}_1 \subseteq \mathcal{A}_2$, $\beta_D(\mathcal{A}_1) \leq \beta_D(\mathcal{A}_2)$. Therefore, $\beta_D(n)$ and $\beta_D(n,k)$ are monotone with respect to n.

However, $\beta_D(n,k)$ is not monotone with respect to k: $\beta_{[\pm 1]}(n,2) = \lceil \log n \rceil$ (see Corollary 2.2.1), $\beta_{[\pm 1]}(n, \frac{n}{2}) = \Omega(\sqrt{n})$ (From Observation 3.9) whereas $\beta_{[\pm 1]}(n, n-2) = 3$ (see Proposition 3.16).

For a family $\mathcal{A} = \{A^1, \ldots, A^m\}$ on $[n]$, and a set $S \subseteq [n]$, the family $\mathcal{A}_{|S} = \{A^1_{|S}, \ldots, A^m_{|S}\}$ is called a family induced by S on $\mathcal{A}$ if $A^j_{|S} = A^j \cap S$, for $1 \le j \le m$. A property $\mathcal{P}$ is *hereditary* if $\mathcal{A} \in \mathcal{P}$ implies $\mathcal{A}_{|S} \in \mathcal{P}$ for every induced family $\mathcal{A}_{|S}$ of $\mathcal{A}$, $S \subseteq [n]$. We note that for any integer t, "$\beta_D(\mathcal{A}) \le t$" is not a hereditary property. This can be demonstrated with the following example. Let $\mathcal{A} = \{\{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}$ be a family on $\{1, \ldots, 5\}$ and $S = \{1,2,3\}$. $\mathcal{A}_{|S} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$ is the subfamily of $\mathcal{A}$ induced by S. It is easy to see that when $D = [\pm 1]$, $\beta_D(\mathcal{A}) = 1$ whereas $\beta_D(\mathcal{A}_{|S}) = 2$.

For any $A, B \subseteq [n]$, if B bisects A, by definition, $[n] \setminus B$ also bisects A. We note this fact in the observation below.

Observation 3.8 *Let $\mathcal{A}$ be a family of subsets of $[n]$ and $\mathcal{B} = \{S_1, \ldots, S_r\}$ be a D-secting family for $\mathcal{A}$, $r \in \mathbb{N}$ and $D = [\pm i]$. Then, $\mathcal{H} = \{H_1, \ldots, H_r\}$ is also a D-secting family for $\mathcal{A}$, where $H_i \in \{[n] \setminus S_i, S_i\}$, $1 \le i \le r$.*

For the rest of the section, assume that n is even (since it does not effect the asymptotics). Note that when k is even (resp., odd), the maximum number of k-sized sets $A \in \mathcal{A}$ that can be bisected with any set $B \subseteq [n]$ is $\binom{\frac{n}{2}}{\frac{k}{2}}^2$ (respectively, $2\binom{\frac{n}{2}}{\lceil \frac{k}{2} \rceil}\binom{\frac{n}{2}}{\lfloor \frac{k}{2} \rfloor}$), $k \in [n]$. This gives a trivial lower bound for $\beta_{[\pm 1]}(n,k)$ using Stirling's approximation, i.e., $\sqrt{2\pi n}(\frac{n}{e})^n \le n! \le e\sqrt{n}(\frac{n}{e})^n$.

Observation 3.9

$$\beta_{[\pm 1]}(n,k) \ge \frac{\binom{n}{k}}{2\binom{\frac{n}{2}}{\lceil \frac{k}{2} \rceil}\binom{\frac{n}{2}}{\lfloor \frac{k}{2} \rfloor}} = \Omega(\sqrt{\frac{k(n-k)}{n}}). \tag{3.1}$$

The constant in the lower bound is $C = \frac{\sqrt{2}\pi^{2.5}}{e^4} \ge .45$. When $k = \frac{n}{2}$, this corresponds to a lower bound of $\Omega(\sqrt{n})$ for $\beta_{[\pm 1]}(n, \frac{n}{2})$. Moreover, using the monotone property,

$\beta_{[\pm 1]}(n) \geq \beta_{[\pm 1]}(n, \frac{n}{2}) = \Omega(\sqrt{n})$. In what follows, we derive improved upper bounds and lower bounds for $\beta_D(n)$. We start our discussion with the case $D = [\pm i]$, $i \in [n]$, followed by the case $D = \{i\}$.

3.2 Bounds for $\beta_{[\pm i]}(n)$

Recall that $\beta_{[\pm i]}(n)$ is the maximum of $\beta_{[\pm i]}(\mathcal{A})$ over all families $\mathcal{A}$ on $[n]$, where $\beta_{[\pm i]}(\mathcal{A})$ denotes the minimum cardinality of a $[\pm i]$-secting family for $\mathcal{A}$.

3.2.1 Upper bounds

Lemma 3.10 $\beta_{[\pm i]}(n) \leq \lceil \frac{n}{2i} \rceil$.

Proof. Let $\mathcal{A}$ denotes the family consisting of all the non-empty subsets of $[n]$. In what follows, we demonstrate a construction that yields a $[\pm i]$-secting family of cardinality $\frac{n}{2i}$ for $\mathcal{A}$, assuming $2i$ divides n. Let $B_1 = \{1, 2, \ldots, \frac{n}{2}\}$. The set B_2 is obtained from B_1 by swapping the largest i elements of B_1 with the smallest i elements in $\overline{B_1}$. So, $B_2 = \{1, 2, \ldots, \frac{n}{2} - i, \frac{n}{2} + i, \frac{n}{2} + i - 1, \ldots, \frac{n}{2} + 1\}$ (we write the swapped elements in descending order for convenience). In general, B_{j+1} is obtained from B_j by swapping the largest i elements of $B_1 \cap B_j$ (i.e., $\{\frac{n}{2} - ij + 1, \ldots, \frac{n}{2} - ij + i\}$) with the smallest i elements of $\overline{B_1} \cap \overline{B_j}$ (i.e., $\{\frac{n}{2} + ij - i + 1, \ldots, \frac{n}{2} + ij\}$). We stop the process at $B_{\frac{n}{2i}} = \{1, \ldots, i, n - i, n - (i-1), \ldots, \frac{n}{2} + 1\}$. Let $\mathcal{B} = \{B_1, \ldots, B_{\frac{n}{2i}}\}$. We illustrate the entire procedure through an example in Figure 3.2 for $n = 12$ and $D = \{-1, 0, 1\}$.

We prove that $\mathcal{B}$ is indeed a $[\pm i]$-secting family for $\mathcal{A}$. For the sake of contradiction, we assume that there exists some $A \subseteq [n]$ such that $|A \cap B_j| - |A \cap \overline{B_j}| \notin D$, for all $B_j \in \mathcal{B}$. Let $c_j := |A \cap B_j| - |A \cap \overline{B_j}|$, $1 \leq j \leq \frac{n}{2i}$. From the construction of B_{j+1} from B_j, observe that $|c_j - c_{j+1}| \leq |B_j \triangle B_{j+1}| = 2i$, $1 \leq j \leq \frac{n}{2i} - 1$. Clearly, $c_1 = d$, for some $d \notin \{-i, \ldots, i\}$.

Claim 1 $c_{\frac{n}{2i}} \leq -d + 2i$ *for* $d > 0$ *(resp.* $\geq -d - 2i$ *for* $d < 0$*).*

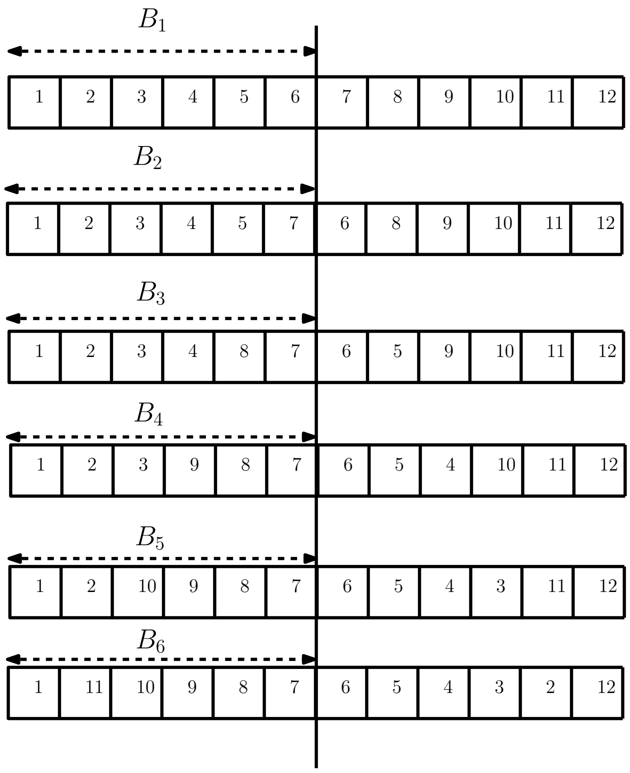

Figure 3.2: $B_1 = \{1, \dots, 6\}$, $B_2 = \{1, \dots, 5, 7\}$, $B_3 = \{1, \dots, 4, 7, 8\}$, $B_4 = \{1, \dots, 3, 7, 8, 9\}$, $B_5 = \{1, \dots, 2, 7, 8, 9, 10\}$, $B_6 = \{1, 7, 8, 9, 10, 11\}$. $\mathcal{B} = \{B_1, \dots, B_6\}$ forms a bisecting family for $2^{[12]}$.

Proof. Let $B_{\frac{n}{2i}+1}$ be the set obtained from $B_{\frac{n}{2i}}$ by swapping the largest i elements $\{1, \dots, i\}$ of $B_1 \cap B_{\frac{n}{2i}}$ with the smallest i elements $\{n - i + 1, \dots, n\}$ of $\overline{B_1} \cap \overline{B_{\frac{n}{2i}}}$. Let $c_{\frac{n}{2i}+1} = |A \cap B_{\frac{n}{2i}+1}| - |A \cap \overline{B_{\frac{n}{2i}+1}}|$. Observe that since $c_1 = d$ and $B_{\frac{n}{2i}+1}$ is $\overline{B_1}$, $c_{\frac{n}{2i}+1} = -d$. Moreover, $|c_{\frac{n}{2i}} - c_{\frac{n}{2i}+1}| \leq 2i$. So, $c_{\frac{n}{2i}}$ is at most $-d + 2i$. The proof for the case of $d < 0$ is similar. □

We now have these exhaustive cases.

1. $d \geq 2i$ (or $d \leq -2i$): Note that $D = \{-i, \dots, +i\}$ and $|c_j - c_{j+1}| \leq 2i$, for all $1 \leq j \leq \frac{n}{2i} - 1$. Using Claim 1, $c_{\frac{n}{2i}} \leq 0$ (resp., $c_{\frac{n}{2i}} \geq 0$). Therefore, there exists at least one index l, $1 \leq l \leq \frac{n}{2i} - 1$, such that $c_l \cdot c_{l+1} \leq 0$. Observe that either of c_l or c_{l+1}, or both lie in $\{-i, \dots, +i\}$. This is a contradiction to our assumption that A is not D-sected by $\mathcal{B}$.

2. $i < d < 2i$: From Claim 1, it is clear that $c_{\frac{n}{2i}} < i$. So, if there exists an index l, $1 \leq l \leq \frac{n}{2i} - 1$, such that $c_l \cdot c_{l+1} \leq 0$, either c_l or c_{l+1} or both lie in $\{-i, \dots, +i\}$.

Otherwise, $c_{\frac{n}{2i}} \in \{0, \dots, i-1\} \subset D$ as desired.

3. $-2i < d < -i$: Similar to the previous case.

This establishes that $\beta_{[\pm i]}(n)$ is at most $\frac{n}{2i}$, when $2i$ divides n. Note that when n is not divisible by $2i$, we can construct $\mathcal{B}$ of cardinality $\lceil \frac{n}{2i} \rceil$ with the same procedure, where $B_{\lceil \frac{n}{2i} \rceil} = \{1, \dots, p, n-p, n-(p-1), \dots, \frac{n}{2}+1\}, p = n \mod 2i$. This completes the proof of Lemma 3.10. □

3.2.2 Lower bounds

To obtain a lower bound for $\beta_D(n)$, it is natural to remove 1 or 2 points from $[n]$ and to proceed with induction. However, we note that, even when $D = \{-1, 0, 1\}$, such a direct induction only yields a lower bound of $\log n$, which is not useful (since we already have a lower bound of $\Omega(\sqrt{n})$ from Section 3.1.3). In order to derive a tight lower bound for $\beta_D(n)$, we use the vector representations of sets and a polynomial representation of Boolean functions.

For any subset $A \subseteq [n]$, let (i) $X_A = (x_1, \dots, x_n) \in \{0,1\}^n$ be the incidence vector such that $x_i = 1$ if and only if $i \in A$; and, (ii)$Y_A = (y_1, \dots, y_n) \in \{-1,1\}^n$ be the incidence vector such that $y_i = 1$ if and only if $i \in A$. Observe that for any two subsets A and B of $[n]$, the dot product of $X_A = (x_1, \dots, x_n)$ with $Y_B = (y_1, \dots, y_n)$, denoted by $\langle X_A, Y_B \rangle$, is equivalent to $|A \cap B| - |A \cap \overline{B}|$. For an even (resp., odd) cardinality subset $A \in \mathcal{A}$, note that the corresponding incidence vector $X_A = (x_1, \dots, x_n)$ is even (resp., odd). Let $\mathcal{A}$ be a family of subsets of $[n]$. Observe that for any even subset $A_e \in \mathcal{A}$ and any arbitrary subset $B \subseteq [n]$, $\langle X_{A_e}, Y_B \rangle \equiv 0 \pmod 2$, i.e., $\langle X_{A_e}, Y_B \rangle \in \{0, \pm 2, \pm 4, \dots\}$. Moreover, for any odd subset $A_o \in \mathcal{A}$, $\langle X_{A_o}, Y_B \rangle \equiv 1 \pmod 2$, i.e., $\langle X_{A_o}, Y_B \rangle \in \{\pm 1, \pm 3, \pm 5, \dots\}$.

We demonstrate that the polynomial representation of Boolean functions [56, 65] is useful to establish lower bounds for $\beta_D(n)$. Let $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function on n variables,say $y_1, \dots, y_n$. For instance, the *parity* function on n variables is simply equal to the monomial $\prod_{j=1}^{n} y_j$. See Section 2.2.1 for definitions and notions related to $sign$ function, multilinear polynomials, representation and weak representation of boolean functions with multilinear polynomials. We have the following result by Minsky and Papert in [49].

Lemma 3.11 *The weak degree of the parity function on n variables is n.*

See Section 2.2.1 for a proof of the above lemma. In what follows, we use the notion of weak degree of the parity function to establish Theorem 3.1.

Lemma 3.12 $\beta_{[\pm i]}(n) \geq \lceil \frac{n}{2i} \rceil$.

Proof. Let $\mathcal{A}$ denote the $2^n - 1$ non-empty subsets of $[n]$. Let $\mathcal{B}$ be a minimum cardinality $[\pm i]$-secting family for $\mathcal{A}$. Let $\mathcal{Y}$ be set of incidence vectors of sets in $\mathcal{B}$, where $Y_B \in \mathcal{Y}$ is $(-1,+1)$-incidence vector corresponding to $B \in \mathcal{B}$. We start the analysis assuming i is even and $i > 0$, and then extend to odd i. For every odd set $A_o \in \mathcal{A}$, there exists a vector $Y \in \mathcal{Y}$ such that $\langle X_{A_o}, Y\rangle - d = 0$, for some $d \in \{-i+1, -i+3, \ldots, i-1\}$. Let $X = (x_1, \ldots, x_n) \in \{0,1\}^n$. We use X_A to denote the incidence vector of any arbitrary set $A \in \mathcal{A}$. Consider the polynomial M on $X = (x_1, \ldots, x_n)$ as

$$M(X) = \left(\prod_{Y_B \in \mathcal{Y}} \left((\langle X, Y_B\rangle)^2 - 1^2 \right) \prod_{Y_B \in \mathcal{Y}} \left((\langle X, Y_B\rangle)^2 - 3^2 \right) \cdots \prod_{Y_B \in \mathcal{Y}} \left((\langle X, Y_B\rangle)^2 - (i-1)^2 \right) \right)^2.$$

From the definitions of $\mathcal{Y}$ and M, it is clear that $M(X)$ is (i) zero when $X = X_{A_o}$ for all odd subsets $A_o \in \mathcal{A}$; and (ii) positive when $X = X_{A_e}$ for all even subsets $A_e \in \mathcal{A}$.

Domain conversion and multilinearization

Recall that a vector $T \in \{0,1\}^n$ is even if the number of 1's in T is even and a vector $T \in \{-1,1\}^n$ is even if the number of -1's in T is even. Consider the polynomial N on $Y = (y_1, \ldots, y_n)$, where each $y_i = \pm 1$.

$$N(y_1,\ldots,y_n) = M(x_1,\ldots,x_n), \tag{3.2}$$

where $x_j = \frac{1-y_j}{2}$, $1 \leq j \leq n$. Note that if $y_i = -1$ (resp. 1), then $\frac{1-y_i}{2}$ becomes 1 (resp. 0). So, if some vector $Y = (y_1,\ldots,y_n)$ includes an even number of -1's, then the vector $(\frac{1-y_1}{2},\ldots,\frac{1-y_n}{2})$ has an even number of 1's, i.e., the reduction of the vector $(y_1,\ldots,y_n)$ from the $\{-1,1\}^n$ domain to $(\frac{1-y_1}{2},\ldots,\frac{1-y_n}{2})$ in the $\{0,1\}^n$ domain preserves the definition of *evenness*. Note that (i) $N(Y)$ evaluates to zero, when $Y = Y_{A_o} \in \{-1,1\}^n$ for all odd subsets $A_o \in \mathcal{A}$; (ii) $sign(N(Y)) = sign(parity(Y))$, when $Y = Y_{A_e} \in \{-1,1\}^n$ for all even subsets $A_e \in \mathcal{A}$. Let $N'(Y = (y_1,\ldots,y_n))$ be the multilinear polynomial obtained from $N(Y = (y_1,\ldots,y_n))$ by repeatedly replacing each y_i^2 in the monomials by 1. $deg(N'(Y)) \leq deg(N(Y))$ and $N'(Y) = N(Y)$, for vectors $Y \in \{-1,1\}^n$.

Clearly, $N'(Y)$ weakly represents the parity function. Each term $(\prod_{Y_B\in\mathcal{Y}}((\langle X_A, Y_B\rangle)^2 - j^2))^2$, $j \in \{1,\ldots,(i-1)\}$, contributes a degree of $4|\mathcal{Y}|$ to the degree of $M(X_A)$, and, there are $\frac{i}{2}$ such terms. Therefore, the degree of $M(X_A)$ is $2|\mathcal{Y}|i$. Moreover, from Equation 3.2, $deg(N'(Y)) \leq deg(N(Y)) = deg(M(X))$. However, from Lemma 3.11, $deg(N'(Y)) \geq n$, which implies $\beta_{[\pm i]}(n) = |\mathcal{Y}| \geq \frac{n}{2i}$.

If $i > 1$ is odd, $M(X)$ is defined as

$$M(X) = \prod_{Y_B\in\mathcal{Y}} \left((\langle X, Y_B\rangle)^2\right) \left(\prod_{Y_B\in\mathcal{Y}} \left((\langle X, Y_B\rangle)^2 - 2^2\right) \prod_{Y_B\in\mathcal{Y}} \left((\langle X, Y_B\rangle)^2 - 4^2\right) \cdots \prod_{Y_B\in\mathcal{Y}} \left((\langle X, Y_B\rangle)^2 - (i-1)^2\right) \right)^2 .$$

Observe that $M(X)$ vanishes for all even vectors and is positive for all odd vectors. The polynomial N on $Y = (y_1,\ldots,y_n)$, where each $y_i = \pm 1$, is now defined as

$$N(y_1,\ldots,y_n) = -M(x_1,\ldots,x_n), \tag{3.3}$$

where $x_j = \frac{1-y_j}{2}$, $1 \leq j \leq n$. Note that degree of $M(X)$ is $2|\mathcal{Y}| + 4|\mathcal{Y}|\frac{i-1}{2} = 2|\mathcal{Y}|i$ and the rest of the arguments are same as the previous case.

We are only left with the cases when $i = 0$ and $i = 1$. Observe that $\beta_D(n)$ for the case of $D = \{0\}$ and $D = \{-1, 0, 1\}$ is same: any bisecting family for a family $\mathcal{A}_1$ consisting of only the $2^{n-1} - 1$ non-empty even subsets of $[n]$ must bisect all the $2^n - 1$ subsets of $[n]$. In this case, take $M(X) = \prod_{Y_B \in \mathcal{Y}} ((\langle X, Y_B \rangle)^2)$ and proceed as before to get $\beta_{[\pm 1]}(n) \geq \frac{n}{2}$.

□

Proof of Theorem 3.1

Statement of Theorem 3.1. $\beta_{[\pm i]}(n) = \lceil \frac{n}{2i} \rceil$, $n \in \mathbb{N}$, $i \in [n]$.

Proof. The proof follows from Lemmas 3.10 and 3.12. □

Let $\mathcal{A}$ consists of $2^n - 1$ non-empty subsets of $[n]$. Then, Theorem 3.1 asserts that the construction of $[\pm i]$-secting family of cardinality $\lceil \frac{n}{2i} \rceil$ in Section 3.2.1 is indeed optimal. Moreover, Theorem 3.1 implies that if we allow the imbalances of intersections up to $\sqrt{n}$, i.e., $D = [\pm\sqrt{n}]$, then a family $\mathcal{B}$ of cardinality $\frac{\sqrt{n}}{2}$ is necessary and sufficient for $\mathcal{A}$.

Corollary 3.12.1 *For $D = [\pm\sqrt{n}]$, $n \in \mathbb{N}$, $\beta_D(n) = \lceil \frac{\sqrt{n}}{2} \rceil$.*

Theorem 3.1 in hindsight establishes that it is not easy to obtain bisecting families without any restriction on structure of the collection of subsets. So, it is natural to ask the following question - "is it possible to obtain small D-secting families for families having relatively small number of hyperedges ideally allowing small value of $|D|$?". For instance, what is the minimum cardinality of a D-secting family provided we have only $O(n)$ sets in $\mathcal{A}$ when $D = [\pm\sqrt{n}]$. In what follows, we demonstrate that D-secting families of cardinality much smaller than $\frac{\sqrt{n}}{2}$ can be computed when $|\mathcal{A}|$ is small. In particular, when $|\mathcal{A}| = n$, we show that only $\log n$ sets in the D-secting family are needed provided $D = [\pm\sqrt{n}]$.

3.2.3 Computing $\beta_{[\pm i]}(\mathcal{A})$ for arbitrary families

In Section 3.1, we discussed about the discrepancy interpretation of the bisection problems. Probabilistic method is an useful tool in computing low discrepancy colorings. The following Chernoff's bound is used extensively to establish upper bounds on the discrepancy of hypergraphs.

Lemma 3.13 *[16] If $X = \sum_{i=1}^{n} X_i$ is the sum of n independent random variables distributed uniformly over $\{-1, 1\}$, then for any $\Delta > 0$,*

$$P[|X| > \Delta] \leq 2e^{-\frac{\Delta^2}{2n}}.$$

In what follows, we obtain an upper bound on $\beta_{[\pm i]}(\mathcal{A})$, when $\mathcal{A}$ is a family of arbitrary sized subsets, with a simple application of Lemma 3.13.

Proof of Theorem 3.2

Statement of Theorem 3.2. Let $\mathcal{A}$ be a family of subsets of $[n]$ and let $|\mathcal{A}| = m$. Let $D = [\pm i]$, where $i = \sqrt{\frac{3n\ln(2m)}{t}}$ and $t \leq \frac{1}{2}\log m$. Then, $\beta_D(\mathcal{A}) \leq t$.

Proof. We pick a set $\mathcal{B}$ of t random subsets $\{B_1, \ldots, B_t\}$ of $[n]$, where for each j, $1 \leq j \leq t$, a point $a \in [n]$ is chosen independently and uniformly at random into B_j. Let $Y_{B_j} = (y_1, \ldots, y_n) \in \{-1, 1\}^n$ be the incidence vector corresponding to B_j: y_i is 1 if and only if $i \in B_j$. For any subset $A \in \mathcal{A}$, $|A \cap B_j| - |A \cap \overline{B_j}|$ can be viewed as sum of $|A|$ random variables distributed uniformly over $\{-1, 1\}$. We say a subset $A \in \mathcal{A}$ is bad with respect to subset $B_j \in \mathcal{B}$ if $||A \cap B_j| - |A \cap \overline{B_j}|| > \sqrt{\frac{3|A|\ln(2m)}{t}}$. Using Chernoff's bound, the probability that a subset $A \in \mathcal{A}$ is bad with respect to a random subset $B_j \in \mathcal{B}$ is

$$P\left[||A \cap B_j| - |A \cap \overline{B_j}|| > \sqrt{\frac{3|A|\ln(2m)}{t}}\right] \leq 2e^{-\frac{3|A|\ln(2m)}{2t|A|}} = 2(\frac{1}{2m})^{\frac{3}{2t}}.$$

Any subset A is bad with respect to $\mathcal{B}$ if $||A \cap B_j| - |A \cap \overline{B_j}|| > \sqrt{\frac{3|A|\ln(2m)}{t}}$, for all $B_j \in \mathcal{B}$. So, A is bad with respect to $\mathcal{B}$ with probability at most $2^t(\frac{1}{2m})^{\frac{3t}{2t}} = \frac{2^{t-1.5}}{m^{1.5}}$. Using union bound, the probability that some subset in $\mathcal{A}$ is bad with respect to $\mathcal{B}$ is at most $m\frac{2^{t-1.5}}{m^{1.5}}$. So, if $2^t \leq \sqrt{m}$ (i.e., $t \leq \frac{1}{2}\log m$), the probability that any subset in $\mathcal{A}$ is bad with respect to $\mathcal{B}$ is at most $\frac{1}{2\sqrt{2}}$. Since the failure probability is less than $\frac{1}{2}$, in expected two iterations, we can obtain a family $\mathcal{B}$ of t subsets such that for every $A \in \mathcal{A}$, there is an $B_j \in \mathcal{B}$ with $||A \cap B_j| - |A \cap \overline{B_j}|| \leq \sqrt{\frac{3n\ln(2m)}{t}}$. $\square$

Note that if $i \geq \sqrt{4.2n+1}$ and $|\mathcal{A}| = O(n^c)$, $c \in \mathbb{N}$, a D-secting family for $\mathcal{A}$ of cardinality $O(\log n)$ can be computed as discussed above (replacing t in the expression of i with $\frac{1}{2}\log m$ and converting the $\ln$ to $\log$ in the numerator). Note that this yields D-secting families of size much smaller than that guaranteed by Corollary 3.12.1 for $\mathcal{A}$ provided $|\mathcal{A}|$ is polynomial in n.

3.3 Bounds for $\beta_i(n)$

In Section 3.2, we established tight bounds for $\beta_D(n)$ when $D = [\pm i]$. In this section, we study $\beta_D(n)$, when D is a singleton set, i.e., $D = \{i\}$. Note that $\beta_i(n) = \beta_{-i}(n)$. Moreover, when $D = \{-i, i\}$, note that $\beta_{\pm i}(n) \leq \beta_i(n) \leq 2\beta_{\pm i}(n)$. Therefore, we focus on establishing bounds for $\beta_i(n)$.

3.3.1 Tight bounds for $\beta_1(n)$

Theorem 3.14 $\beta_1(n) = \lceil \frac{n}{2} \rceil$, $n \in \mathbb{N}$.

Proof. As mentioned in Section 3.1, when $D = \{1\}$, the family $\mathcal{A}$ should consist of all the odd subsets of $[n]$. Let $\mathcal{Y}$ be a minimum sized set of $\{-1,+1\}^n$ vectors such that for every odd set $A_o \in \mathcal{A}$, there exists a vector $Y_B \in \mathcal{Y}$ such that $\langle A_o, Y_B \rangle - 1 = 0$. Consider the polynomial M on $X = (x_1, \ldots, x_n)$.

$$M(X) = \prod_{Y_B \in \mathcal{Y}} (\langle X, Y_B \rangle - 1)^2 \tag{3.4}$$

Note that if $N'(Y)$ is obtained from $M(X)$ after domain conversion and multilinearization, N' weakly represents the parity function. Using Lemma 3.11, $deg(M(X)) = 2|\mathcal{Y}| \geq deg(N'(Y)) \geq n$ and therefore $|\mathcal{Y}| \geq \lceil \frac{n}{2} \rceil$. In what follows, we demonstrate a construction of a family $\mathcal{B}$ of cardinality $\lceil \frac{n}{2} \rceil$ such that for every odd subset $A \in \mathcal{A}$, there exists some $B \in \mathcal{B}$ with $|A \cap B| - |A \cap \overline{B}| = 1$.

Consider the family $\mathcal{A}$ consisting of all the odd subsets of $[n]$. Consider the case when n is even; the odd case is similar except the ceilings in the final expression. Note that if $n \leq 2$, we can choose $\mathcal{B} = \{\{1, 2\}\}$ to get the desired intersection property. So, we consider the case when $n \geq 4$. Let $B_1 = \{1, 2, \ldots, \frac{n}{2} + 1\}$. B_2 is obtained from B_1 by swapping $\{\frac{n}{2} + 1\}$ with $\{\frac{n}{2} + 2\}$, i.e., $B_2 = \{1, 2, \ldots, \frac{n}{2}, \frac{n}{2} + 2\}$. In general, B_{j+1} is obtained from B_j by replacing the point $\frac{n}{2} - j + 2$ with $\frac{n}{2} + j + 1$. We stop the process at $B_{\frac{n}{2}} = \{1, 2, n, n-1, \ldots, \frac{n}{2} + 2\}$. Let $\mathcal{B} = \{B_1, \ldots, B_{\frac{n}{2}}\}$.

Claim 2 *(i) For any odd subset $A_o \subseteq \{3, \ldots, n\}$, there exists some B_j and B_l in $\mathcal{B}$ such that $|A \cap B_j| = \lceil \frac{|A|}{2} \rceil$, and $|A \cap B_l| = \lfloor \frac{|A|}{2} \rfloor$, and (ii) For any even subset $A_e \subseteq \{3, \ldots, n\}$, there exists some B_j in $\mathcal{B}$ such that $|A \cap B_j| = \frac{|A|}{2}$.*

To see the correctness of the claim, consider an arbitrary set A, $A \subseteq \{3, \ldots, n\}$, such that $|A \cap B_1| - |A \cap \overline{B_1}| = d$, for some $d \in \mathbb{N} \setminus 0$. Then, it follows from the construction that $|A \cap B_{\frac{n}{2}}| - |A \cap \overline{B_{\frac{n}{2}}}| = -d$. Observe that for any j, $1 \leq j \leq \frac{n}{2} - 1$, the difference between $|A \cap B_{j+1}| - |A \cap \overline{B_{j+1}}|$ and $|A \cap B_j| - |A \cap \overline{B_j}|$ is either -2, 0 or 2. So, the claim follows.

Now, to complete the proof, we need to consider the following exhaustive case for an odd subset A_o.

1. $A_o \subseteq \{3, \ldots, n\}$: A_o has the desired intersection property using Claim 2.

2. $|A_o \cap \{3, \ldots, n\}| = |A_o| - 1$: Using Claim 2, there exists some B_j in $\mathcal{B}$ such that the even subset $A_o \cap \{3, \ldots, n\}$ is bisected by B_j. Clearly, $|A_o \cap B_j| = \lceil \frac{|A_o|}{2} \rceil$.

3. $|A_o \cap \{3, \ldots, n\}| = |A_o| - 2$: In this case, $\{1, 2\} \subset A_o$. From Claim 2, there exists some B_j in $\mathcal{B}$ such that $|A'_o \cap B_j| = \lfloor \frac{|A'_o|}{2} \rfloor$, where $A'_o = A_o \cap \{3, \ldots, n\}$.

Then, $|A_o \cap B_j| = \lceil \frac{|A_o|}{2} \rceil$.

This establishes that $\beta_1(n) \leq \lceil \frac{n}{2} \rceil$ and completes the proof of Theorem 3.14. □

3.3.2 Bounds for $\beta_i(n)$, $i \geq 2$

In the following section, we extend the notion of $\beta_1(n)$ to arbitrary values of i. Note that when $i = 0$, $\beta_0(n) = \beta_{[\pm 1]}(n) = \lceil \frac{n}{2} \rceil$ (see Theorem 3.1). The case when $i = 1$ is resolved by Theorem 3.14. We assume that $i \geq 2$ in the remainder of the section.

Proof of Theorem 3.3

Statement of Theorem 3.3. $\frac{n-i+1}{2} \leq \beta_i(n) \leq n - i + 1$, $n \in \mathbb{N}$, $i \in [n]$.

Proof. Let $\mathcal{A}$ consist of all subsets of $[n]$ such that $A \in \mathcal{A}$ if and only if $|A| \equiv i \pmod 2$ and $|A| \geq i$. Let $\mathcal{B} = \{B_1 = [i], B_2 = B_1 \cup \{i+1\}, \ldots, B_{n-i+1} = B_{n-i} \cup \{n\}\}$. Observe that $\mathcal{B}$ is indeed an i-secting family for $\mathcal{A}$. Therefore, $\beta_i(n) \leq n-i+1$. In what follows, we prove the lower bound for $\beta_i(n)$ assuming i to be an even integer greater than 1. The case for odd i can be treated analogously.

We invoke the notion of weak representation of the parity function to establish a lower bound. Let $\mathcal{A}$ denote the $2^n - 1$ non-empty subsets of $[n]$. Let $\mathcal{B}$ be a minimum cardinality i-secting family for $\mathcal{A}$. Let $\mathcal{Y}$ be the set of incidence vectors of sets in $\mathcal{B}$, where each $Y_B \in \mathcal{Y}$ is a $(-1, +1)$-incidence vector corresponding to $B \in \mathcal{B}$. So, for any even subset $A_e \subseteq [n]$ with $|A_e| \geq i$, there exists a vector $Y_B \in \mathcal{Y}$ such that $\langle X_{A_e}, Y_B \rangle - i = 0$, where X_{A_e} is the 0–1 incidence vector of A_e. We define the polynomials P, M and F on $X = (x_1, \ldots, x_n)$ as follows.

$$M(X) = \prod_{Y_B \in \mathcal{Y}} (\langle X, Y_B \rangle - i)^2. \tag{3.5}$$

$$F(X) = \sum_{S \in \binom{[n]}{i-1}} \prod_{j \in S} x_j.$$

$$P(X) = M(X)F(X). \tag{3.6}$$

Observe that (i) $P(X)$ evaluates to zero when $X = X_A$, for all subsets A of size at most $i-2$ (since $F(X)$ vanishes for these subsets), (ii) $P(X)$ evaluates to zero when $X = X_{A_e}$, for all even subsets A_e of size at least i (since $M(X)$ vanishes for these subsets), and, (iii) $P(X)$ is strictly positive when $X = X_{A_o}$, for all odd subsets A_o of size at least $i-1$. Consider the polynomial Q on $Y = (y_1, \ldots, y_n)$, where each $y_j \in \{-1, 1\}$.

$$Q\left(y_1, \ldots, y_n\right) = -P\left(x_1, \ldots, x_n\right) \tag{3.7}$$

where $x_j = \frac{1-y_j}{2}$, $1 \leq j \leq n$. Let $Q'(Y)$ be the multilinear polynomial obtained from $Q(Y)$ by replacing each occurrence of a y_j^2 by 1, repeatedly. Note that (i) $Q'(Y)$ evaluates to zero for even subsets of $[n]$, and (ii) if $Q'(Y)$ is non-zero on some odd subset Y, then $sign(Q'(Y)) = sign(parity(Y))$. Therefore, $Q'(Y)$ weakly represents the parity function. From Lemma 3.11, $Q'(Y)$ has degree at least n, and $deg(P(X)) = (i-1) + 2|\mathcal{Y}| \geq deg(Q'(Y)) \geq n$. So, $|\mathcal{Y}| \geq \frac{n-i+1}{2}$. □

3.4 Bisecting k-uniform families

In this section, we discuss the problem of bisection for k-uniform families. We assume that n is even throughout the remaining part of the chapter. We focus on establishing bounds for $\beta_D(n,k)$ when $D = [\pm 1]$.

3.4.1 Some observations for $\beta_{[\pm 1]}(n,k)$

Observation 3.15 *Let n be an even integer and $\mathcal{B}$ be an optimal bisecting family for a family $\mathcal{A} = \binom{[n]}{k}$ such that each subset $B \in \mathcal{B}$ has cardinality $\frac{n}{2}$. Then, $\beta_{[\pm 1]}(n, n-k) \leq \beta_{[\pm 1]}(n,k)$*

Proof. It is not hard to see that the bisecting family $\mathcal{B}$ for $\mathcal{A}$ is also a bisecting family for $\overline{\mathcal{A}} = \binom{[n]}{n-k}$ when n is even and each subset in $\mathcal{B}$ is a part of an equal-sized bipartition of n. □

From Corollary 2.2.1, we know that $\beta_{[\pm 1]}(n, 2) = \lceil \log n \rceil$. Moreover, when n is of the form 2^t, for some $t \in \mathbb{N}$, we can obtain a bisecting family $\mathcal{B} = \{A_1, \ldots, A_{\log n}\}$ for the family $\mathcal{A} = \binom{[n]}{2}$ in the following way. (i) For $j \in [n]$, obtain the $\log n$ bit binary code equivalent to $j - 1$ and assign it to j. (ii) Elements with l-th bit as 1 form the set A_l. Using Corollary 2.2.1, $\mathcal{B}$ is an optimal bisecting family for $\mathcal{A}$, and $|A_l| = \frac{n}{2}$, for all $A_l \in \mathcal{B}$. Using Observation 3.15, it follows that $\beta_{[\pm 1]}(n, n-2) \leq \log n$, when n is a power of 2. However, when the difference between n and k is a small constant, we can achieve much better bounds for $\beta_{[\pm 1]}(n, k)$ as follows.

Proof of Theorem 3.7

Statement of Theorem 3.7. Let $\mathcal{A} = \binom{[n]}{k} \cup \binom{[n]}{k+1} \ldots \cup \binom{[n]}{n}$. Then, $\frac{n-k+1}{2} \leq \beta_{[\pm 1]}(\mathcal{A}) \leq \min\{\frac{n}{2}, n - k + 1\}$.

Proof. The upper bound of $\frac{n}{2}$ follows from Lemma 3.10. Let $x = n - k$. We obtain a bisecting family for $\mathcal{A}$ of cardinality $x + 1$ in the following way. Let S and T denote two disjoint $\lceil \frac{k}{2} \rceil$ and $\lfloor \frac{k}{2} \rfloor$ elements subset of $[n]$, respectively. Let $c_1, \ldots, c_x$ denote the remaining elements of $[n]$. Let $S_0 = S$, and for any $j \in [x]$, $S_j = S_{j-1} \cup \{c_j\}$. Let $\mathcal{B} = \{S_0, \ldots, S_x\}$. We claim that $\mathcal{B}$ is a bisecting family for $\mathcal{A}$. For any set A of cardinality k', $k \leq k' \leq n$, that is not bisected by S_0, $|A \cap S_0| < \frac{k'}{2}$ and $|A \cap S_x| \geq \frac{k'}{2}$. The upper bound follows from the observation that $|A \cap S_{j+1}|$ differs from $|A \cap S_j|$ by at most 1.

The proof of the lower bound $\frac{n-k+1}{2}$ for $\beta_{[\pm 1]}(\mathcal{A})$ is in the same spirit as the proof of the lower bound of Theorem 3.3; we give the proof for completeness. We assume that $k \geq 2$ and is even; the case when k is odd is analogous. Let $\mathcal{B}$ be a minimum cardinality $[\pm 1]$-secting family for $\mathcal{A}$. Let $\mathcal{Y}$ be the set of incidence vectors of sets in $\mathcal{B}$, where each vector $Y_B \in \mathcal{Y}$ is a $(-1, +1)$-vector corresponding to $B \in \mathcal{B}$. We define the polynomials P, M and F on $X = (x_1, \ldots, x_n)$ as follows.

$$M(X) = \prod_{Y_B \in \mathcal{Y}} (\langle X, Y_B \rangle)^2 \text{ (note the difference from Equation 3.5).} \tag{3.8}$$

$$F(X) = \sum_{S \in \binom{[n]}{k-1}} \prod_{j \in S} x_j. \tag{3.9}$$

$$P(X) = M(X)F(X). \tag{3.10}$$

Observe that (i) $P(X)$ evaluates to zero when $X = X_A$, for all subsets A of size at most $k-2$ (since $F(X)$ vanishes for these subsets), (ii) $P(X)$ evaluates to zero when $X = X_{A_e}$, for all even subsets A_e of size at least k (since $M(X)$ vanishes for these subsets), and, (iii) $P(X)$ is strictly positive when $X = X_{A_o}$, for all odd subsets A_o of size at least $k-1$. Note that if $Q'(Y)$ is obtained from $P(X)$ after domain conversion and multilinearization, $Q'(Y)$ weakly represents the parity function. From Lemma 3.11, $Q'(Y)$ has degree at least n, and $deg(P(X)) = (k-1) + 2|\mathcal{Y}| \geq deg(Q'(Y)) \geq n$. So, $|\mathcal{Y}| \geq \frac{n-k+1}{2}$. □

Note that using Theorem 3.7 for $k = n-2$, we get, $\beta_{[\pm 1]}(n, n-2) \leq 3$. This is surprising since (i) $\mathcal{A} = \binom{[n]}{n-2}$ has the same number of subsets as $\overline{\mathcal{A}} = \binom{[n]}{2}$, (ii) the maximum number of sets of $\mathcal{A}$ and $\overline{\mathcal{A}}$ that can be bisected by a single set $B \in \mathcal{B}$ is $(\frac{n}{2})^2$, and (iii) $\beta_0(n, 2) = \lceil \log n \rceil$.

Proposition 3.16 $\beta_{[\pm 1]}(n, n-2) = 3$, *for every even integer* n *greater than 4.*

Proof. We only need to show that $\beta_{[\pm 1]}(n, n-2) > 2$. Note that since the hyperedges are of cardinality $n-2$, every set in an optimal bisecting family $\mathcal{B}$ is of cardinality $\frac{n}{2}-1$, $\frac{n}{2}$, or $\frac{n}{2}+1$. Consider an optimal bisecting family $\mathcal{B} = \{A_1, A_2\}$ of cardinality 2 for $\mathcal{A} = \binom{[n]}{n-2}$. Since $\beta_{[\pm 1]}(n, n-2) \leq 3$, any optimal bisecting family $\mathcal{B}$ for $\mathcal{A}$ must contain at least one set of size other than $\frac{n}{2}$. Otherwise, using Observation 3.15, $\mathcal{B}$ is a bisecting family of cardinality less than $\log n$ for $\binom{[n]}{2}$, a contradiction to Corollary 2.2.1. Without loss of generality, assume that $|A_1| \neq \frac{n}{2}$. Using Observation 3.8, we can also assume that $|A_1| = \frac{n}{2}-1$. The rest of the proof is an exhaustive case analysis based on the cardinality of A_2. Let $A^1 = A_1 \cap A_2$ and $A^2 = A_1 \setminus A_2$.

1. $|A_2| = \frac{n}{2}$. If at least one of A^1 or A^2 is of size at least 2, the $(n-2)$-sized subset missing 2 elements of $[n]$ both from either A^1 or A^2 is not bisected by $\mathcal{B}$. This is the case for any $n \geq 8$. For $n = 6$, if $|A^1| = 1$, the set missing the element in A^1 and an element from $[n] \setminus A_1 \setminus A_2$ is not bisected by $\mathcal{B}$ (for example, when $n = 6$, $A_1 = \{1, 2\}$, $A_2 = \{2, 3, 4\}$, the sets $\{2, 3, 4, 5\}$ and $\{2, 3, 4, 6\}$ are not bisected). For $n = 6$, if $|A^1| = 0$, for example, say $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, then $A^2 = 2$, and we are done.

2. $|A_2| = \frac{n}{2} + 1$. If $|A^2| \geq 2$, the $(n-2)$-sized subset missing 2 elements both from A^2 is not bisected by $\mathcal{B}$. So, $|A^2| \leq 1$. If $A^2 = \{y\}$, then an $(n-2)$-sized subset missing y and one element from A^1 is not bisected by $\mathcal{B}$. If $A^2 = \emptyset$, then any $(n-2)$-sized subset missing one element each from A_1 and $[n] \setminus A_2$ is not bisected by $\mathcal{B}$.

3. $|A_2| = \frac{n}{2} - 1$. Using Observation 3.8, this case is identical to Case 2. $\square$

3.4.2 Proof of Theorem 3.4

Note that the lower bound of $\Omega(\sqrt{\frac{k(n-k)}{n}})$ for $\beta_{[\pm 1]}(n, k)$ is given by Observation 3.9. However, when k is a constant, Observation 3.9 asserts only a $\Omega(\sqrt{k})$ lower bound on $\beta_{[\pm 1]}(n, k)$. An improved lower bound on $\beta_{[\pm 1]}(n, k)$ for constant k given by Theorem 3.4 is proven below.

Statement of Theorem 3.4.

$$\beta_{[\pm 1]}(n, k) \geq \begin{cases} \log(n - k + 2), \text{ when } k \text{ is even and } \frac{k}{2} \text{ is odd,} \\ \lceil (\log \lceil \frac{n}{\lceil \frac{k}{2} \rceil} \rceil) \rceil, \text{ for any } k \geq 2. \end{cases}$$

Proof. We prove the first lower bound given in Theorem 3.4 under the assumption that k is even and $\frac{k}{2}$ is odd. Let $\mathcal{B} = \{B_1, \ldots, B_t\}$ be a bisecting family for the family

$\mathcal{A} = \binom{[n]}{k}$. For every $B_j \in \mathcal{B}$, let $\mathcal{A}_j$ be the collection of k-sized sets that are bisected by B_j. We estimate a lower bound for t. For any family $\mathcal{F} \subseteq \binom{[n]}{k}$, we associate a graph $G(\mathcal{F})$ in the following way:

$$V(G(\mathcal{F})) = \{S \in \binom{[n]}{\frac{k}{2}} : S \subseteq A, A \in \mathcal{F}\}$$
$$E(G(\mathcal{F})) = \{\{S_1, S_2\} : S_1 \cap S_2 = \emptyset, S_1, S_2 \in V(G(\mathcal{F}))\}.$$

Observe that $G(\mathcal{A})$ is the Kneser graph $KG(n, \frac{k}{2})$ (for definitions and results related to Kneser graphs, see [14, 1]). For every k-sized subset $A \in \mathcal{A}$, there are $\binom{k}{\frac{k}{2}}/2$ edges in $E(G(\mathcal{A}))$: an edge between any two disjoint $\frac{k}{2}$ sets. From the definition of $\mathcal{A}_1, \ldots, \mathcal{A}_t$, $\cup_{j=1}^{t} G(\mathcal{A}_j) = G(\mathcal{A})$.

Claim 3 *Each $G(\mathcal{A}_j)$ is a bipartite graph.*

Let $A \in \mathcal{A}_j$. Consider a fixed $\frac{k}{2}$ sized subset S of A. If $|S \cap B_j| > \lfloor \frac{k}{4} \rfloor$, S is placed in the first partite set of $G(\mathcal{A}_j)$; otherwise S is placed in the second partite set of $G(\mathcal{A}_j)$. Note that since $\frac{k}{2}$ is odd, $|S \cap B_j|$ can never be equal to $|S \cap \overline{B_j}|$. It is now easy to see that there is no edge inside the first or second partite set of $G(\mathcal{A}_j)$.

$G(\mathcal{A}_1), \ldots, G(\mathcal{A}_t)$ are bipartite graphs whose union covers $G(\mathcal{A})$. Since $G(\mathcal{A})$ is the Kneser graph $KG(n, \frac{k}{2})$, its chromatic number is $n - k + 2$ (see [46, 1]). So, using Proposition 2.2, we get, $t \geq \lceil \log(n - k + 2) \rceil$ [1]. That is, $\beta_{[\pm 1]}(n, k) \geq \lceil \log(n - k + 2) \rceil$, when k is even and $\frac{k}{2}$ is odd. This concludes the proof of the first lower bound given by Theorem 3.4.

To prove the second lower bound of Theorem 3.4, consider a bisecting family $\mathcal{B} = \{B_1, \ldots, B_t\}$ of $\mathcal{A} = \binom{[n]}{k}$. Observe that for every $\lceil \frac{k}{2} \rceil + 1$-sized set $S \subseteq [n]$, there exists an $B_j \in \mathcal{B}$ such that $S \cap B_j \neq \emptyset$ and $S \cap \overline{B_j} \neq \emptyset$. For every $B_j \in \mathcal{B}$, let $\mathcal{A}_j$ be the collection of $\lceil \frac{k}{2} \rceil + 1$-sized sets that has a non-empty intersection with both B_j and

[1]Note that Proposition 2.2 does not guarantee equality since the $\lceil \log(n - k + 2) \rceil$ bipartite graphs that cover $G(\mathcal{A})$ as per Proposition 2.2 may not correspond to valid $\mathcal{A}_j$'s.

$\overline{B_j}$. Observe that

$$\bigcup_{j=1}^{t} \mathcal{A}_j = \binom{[n]}{\lceil \frac{k}{2} \rceil + 1}. \tag{3.11}$$

Construct hypergraphs $G_1, \ldots, G_t$, where $V(G_j) = [n]$ and $E(G_j) = \mathcal{A}_j$. To each point $v \in [n]$, assign an t length 0–1 bit vector: jth bit is 1 if and only if $v \in B_j$. Color the points in $[n]$ with the decimal equivalent of its bit vector. Let $f : [n] \rightarrow \{0, 1, \ldots, 2^t - 1\}$ denote this coloring. We show that none of the $\binom{[n]}{\lceil \frac{k}{2} \rceil + 1}$ sets remain monochromatic under f. Assume for the sake of contradiction that $S \in \binom{[n]}{\lceil \frac{k}{2} \rceil + 1}$ is monochromatic under f. From Equation 3.11, there exists an $\mathcal{A}_j$ such that $S \in \mathcal{A}_j$. From the definition of $\mathcal{A}_j$, S has non-empty intersection with both B_j and $\overline{B_j}$. Therefore, the jth bits of the t length 0–1 bit vectors of all the points in S cannot be the same. Therefore, S contains at least two points of different color under f, i.e., S is not monochromatic. It is well known that the chromatic number of $\binom{[n]}{\lceil \frac{k}{2} \rceil + 1}$, $\chi(\binom{[n]}{\lceil \frac{k}{2} \rceil + 1})$, is $\lceil \frac{n}{\lceil \frac{k}{2} \rceil} \rceil$. Since f uses 2^t colors, we have, $2^t \geq \lceil \frac{n}{\lceil \frac{k}{2} \rceil} \rceil$ Therefore, $\beta_{[\pm 1]}(n, k) = |\mathcal{B}| = t \geq \lceil (\log \lceil \frac{n}{\lceil \frac{k}{2} \rceil} \rceil) \rceil$.

This completes the proof of Theorem 3.4. □

3.4.3 Proof of Theorem 3.5

We know that $\beta_{[\pm 1]}(n) = \lceil \frac{n}{2} \rceil$ (see Theorem 3.1). The number of $\frac{n}{2}$-sized subsets of $[n]$ that can be bisected by a single subset $B \subseteq [n]$ is at most $2(\binom{\frac{n}{2}}{\frac{n}{4}})^2$. This gives a trivial lower bound of $\Omega(\sqrt{n})$ for $\beta_{[\pm 1]}(n, \frac{n}{2})$. In this section, we prove a stronger result using a theorem of Keevash and Long [39] which is an improvement over a theorem of Frankl and Rödl [29]. Given $q \in \mathbb{N}$, a set $\mathcal{C}$ is called a q-ary code if $\mathcal{C} \subseteq [q]^n$, for $q \geq 2$. For any $x, y \in [q]^n$, the *Hamming distance* between x and y, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, denoted by $d_H(x, y)$, is $|\{i \in [n] : x_i \neq y_i\}|$. For any code $\mathcal{C}$, let $d(\mathcal{C})$ be the set of all the Hamming distances allowed for any $x, y \in \mathcal{C}$. A code is called *d-avoiding* if $d \notin d(\mathcal{C})$. We have the following upper bound on the cardinality of a *d-avoiding* code $\mathcal{C}$ as given in [39].

Theorem 3.17 *[39] Let $\mathcal{C} \subseteq [q]^n$ and let ϵ satisfy $0 < \epsilon < \frac{1}{2}$. Suppose that $\epsilon n < d < (1-\epsilon)n$ and d is even if $q = 2$. If $d \notin d(\mathcal{C})$, then $|\mathcal{C}| \leq q^{(1-\delta)n}$, for some positive constant $\delta = \delta(\epsilon)$.*

In what follows, we prove Theorem 3.5.

Statement of Theorem 3.5. Let c be a constant such that $0 < c < \frac{1}{2}$ and $n \in \mathbb{N}$. If $cn < k < (1-c)n$, then

$$\max\left\{\beta_{[\pm 1]}(n,k), \beta_{[\pm 1]}(n,k-1), \beta_{[\pm 1]}(n,k-2), \beta_{[\pm 1]}(n,k-3)\right\} \geq \delta n,$$

where $\delta = \delta(c)$ is some real positive constant.

Proof. Consider a bisecting family $\mathcal{B} = \{B_1, \dots, B_m\}$ of minimum cardinality for $\binom{[n]}{l}$, where $cn < l < (1-c)n$ is even and $\frac{l}{2}$ is odd, for some constant c, $0 < c < \frac{1}{2}$. Let X_A denote the 0–1 incidence vector corresponding to a set $A \subseteq [n]$. Let V denote the vector space generated by the incidence vectors of $\mathcal{B}$ over $\mathbb{F}_2$. Observe that for any $A \in \binom{[n]}{l}$, there exists an $B \in \mathcal{B}$ such that $|A \cap B| = \frac{l}{2}$. Since $\frac{l}{2}$ is odd, $\langle X_A, X_B \rangle = 1$, i.e., $X_A \notin V^{\perp}$, where $V^{\perp}$ is the subspace of the vector space $\{0,1\}^n$ over $\mathbb{F}_2$ which contains all the vectors perpendicular to V. So, $V^{\perp}$ is a subspace containing no vector of weight l. For any $X_B, X_C \in V^{\perp}$, $X_B + X_C$ has weight $|B \triangle C| \neq l$. Moreover, l is even. Since $cn < l < (1-c)n$, using Theorem 3.17, there exists a positive constant $\delta = \delta(c)$ such that $|V^{\perp}| \leq 2^{n(1-\delta)}$. So, $dim(V^{\perp}) \leq n - \lfloor \delta n \rfloor$. It follows that $dim(V) \geq \lfloor \delta n \rfloor$. To complete the proof of the theorem, note that for any k, there exists an $l \in \{k, k-1, k-2, k-3\}$ such that l is even and $\frac{l}{2}$ is odd. $\square$

3.4.4 $\beta_{[\pm 1]}(n,k)$ and the computation of bisecting families

An important probabilistic tool used in this section is the Lovász local lemma [26]. Let $\mathcal{A}$ be a family of subsets of $[n]$. The *dependency* of a set $A \in \mathcal{A}$ denoted by $d(A, \mathcal{A})$ is the number of subsets $\widehat{A} \in \mathcal{A}$, such that (i) $|A \cap \widehat{A}| \geq 1$, and (ii) $A \neq \widehat{A}$. The *dependency* of a family $\mathcal{A}$, denoted by $d(\mathcal{A})$ or simply d, is the maximum dependency of any subset A in the family $\mathcal{A}$. We have the following corollary of the Lovász local lemma from [51].

Lemma 3.18 *[51] Let $\mathcal{P}$ be a finite set of mutually independent random variables in a probability space. Let $\mathcal{A}$ be a finite set of events determined by these variables, where $m = |\mathcal{A}|$. For any $A \in \mathcal{A}$, let $\Gamma(A)$ denote the set of all the events in $\mathcal{A}$ that depend on A. Let $d = \max_{A \in \mathcal{A}} |\Gamma(A)|$. If $\forall A \in \mathcal{A} : P[A] \leq p$ and $ep(d+1) \leq 1$, then an assignment of the variables not violating any of the events in $\mathcal{A}$ can be computed using expected $\frac{1}{d}$ resamplings per event and expected $\frac{m}{d}$ resamplings in total.*

Proof of Theorem 3.6

Statement of Theorem 3.6. For a family $\mathcal{A}$ consisting of k-sized subsets of $[n]$ and dependency d, $\beta_{[\pm 1]}(\mathcal{A}) \leq \frac{\sqrt{k}}{c}(\ln(d+1)+1)$, where $c = 0.67$.

Proof. Let $\mathcal{A}$ be a family of k-sized subsets of $[n]$, $\mathcal{A} \subseteq \binom{[n]}{k}$, with dependency d. Assume that k is even. Consider a family $\mathcal{B} = \{B_1, \ldots, B_t\}$: each $B_j \in \mathcal{B}$ is a subset of $[n]$ constructed by choosing each point $x \in [n]$ randomly and independently with probability $\frac{1}{2}$ into B_j. Let p be the probability that a fixed subset $A \in \mathcal{A}$ is bisected by some $B_j \in \mathcal{B}$.

$$p = \frac{\binom{k}{\frac{k}{2}}}{\binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k}} \geq \frac{c}{\sqrt{k}}, \text{ where } c = 0.67.$$

So, the probability that A is not bisected by a fixed B_j is $1-p$ which is at most $1 - \frac{c}{\sqrt{k}}$. Therefore, the probability that A is bisected by none of the $B_j \in \mathcal{B}$ is $(1-p)^t$ which is at most $(1 - \frac{c}{\sqrt{k}})^t \leq e^{-\frac{ct}{\sqrt{k}}}$. When $t \geq \frac{\sqrt{k}}{c}(\ln(d+1)+1)$, we get from Lemma 3.18 that there exists a bisecting family of size $\frac{\sqrt{k}}{c}(\ln(d+1)+1)$ for any family $\mathcal{A}$ of k-sized sets, where d denotes the dependency of family $\mathcal{A}$.

In fact, if $\mathcal{A}$ is $\binom{[n]}{k}$ and we choose the subsets $B_j \in \mathcal{B}$ of cardinality exactly $\frac{n}{2}$ uniformly and independently at random from $\binom{[n]}{\frac{n}{2}}$, then $p = \frac{\binom{\frac{n}{2}}{\frac{k}{2}}^2}{\binom{n}{k}} \geq c_1\sqrt{\frac{n}{(n-k)k}}$

($c_1 \geq 0.53$). Therefore, the probability that A is bisected by none of the $B_j \in \mathcal{B}$ is $(1-p)^t$. Using Lemma 3.18, we can compute a bisecting family for $\binom{[n]}{k}$ of size $\frac{1}{c_1}\sqrt{\frac{k(n-k)}{n}}(\ln(d+1)+1)$. Therefore, using Observation 3.9, $\beta_{[\pm 1]}(n,k)$ is $O((\ln(d+1)+1))$-approximable.

The proof for the case when k is odd is similar to the above proof. In fact, we get a small constant factor improvement over the bound given in Theorem 3.6. □

Let $m = |\mathcal{A}|$. Since, $d+1 \leq m \leq \binom{n}{k} < (\frac{en}{k})^k$, we get, $\beta_{[\pm 1]}(n,k) \leq \frac{1}{c_1}\sqrt{\frac{k(n-k)}{n}}(\ln m + 1) \leq \frac{k}{c_1}\sqrt{\frac{k(n-k)}{n}}\ln(\frac{en}{k})$.

3.5 Bisecting families for Hadamard set systems

The discrepancy interpretation of bisecting families leads us to the investigation of $\beta_{[\pm 1]}(\mathcal{A})$ for recursive Hardamard set systems.

Definition 3.19 *A Hadamard matrix H is a $n \times n$ matrix with (i) each entry being either $+1$ or -1, and (ii) any two distinct columns being orthogonal, i.e., $H^T H = nI$, where I is the $n \times n$ identity matrix.*

By convention, the first row and first column of H are all ones. By a recursive construction, $H(k)$ of size $2^k \times 2^k$ can be obtained from $H(k-1)$ of size $2^{k-1} \times 2^{k-1}$ as follows:

$$H(k) = \begin{bmatrix} H(k-1) & H(k-1) \\ H(k-1) & -H(k-1) \end{bmatrix},$$

where $H(0) = 1$. Note that except the first row, every other row of the Hadamard matrix $H(k)$ must contain equal number of 1's and -1's, since the columns are orthogonal and $H(k)$ is symmetric. Let $A = \frac{1}{2}(H(k) + J(k))$, where J is the $2^k \times 2^k$ matrix whose every entry is $+1$. The matrix A corresponds to the Hadamard set system $HF(k)$, where $HF(k) = \{A_1, \ldots, A_{2^k}\}$, and, $j \in A_i$ if and only if the (i,j) entry of A is one. So, from construction, every subset $A_j \in HF(k)$ except A_1 is of cardinality exactly 2^{k-1}.

It is a well known fact that a Hadamard set system HF of order $n \times n$ has a discrepancy at least $\frac{\sqrt{n-1}}{2}$ [48, p. 106]. Therefore, $\beta_{[\pm 1]}(HF(k)) \geq 2$. In what follows, we show that $\beta_{[\pm 1]}(HF(k)) \leq 2$ for all Hadamard set systems obtained from the recursively constructed Hadamard matrix $H(k)$, $k > 1$.

Consider the Hadamard set system $HF(k)$, which is represented by the incidence matrix A. Let $B_1 = \{1, \ldots, 2^{k-1}\}$. Observe that A_1 through $A_{2^{k-1}}$ of $HF(k)$ are bisected by B_1 due to the recursive construction. $A_{2^{k-1}+1}$ represented by the $2^{k-1}+1$th row of A is not bisected by B_1. In fact, $|A_{2^{k-1}+1} \cap B_1| - |A_{2^{k-1}+1} \cap ([2^k] \setminus B_1)| = 2^{k-1}$. The subsets $A_{2^{k-1}+2}$ through A_{2^k} of $HF(k)$ are bisected by B_1 since every row, except the first row, of $H(k-1)$ and $-H(k-1)$ contain equal number of 1's and -1's. $A_{2^{k-1}+1}$ represented by the $2^{k-1}+1$th row of A can be bisected by a second subset $B_2 = \{1, \ldots, 2^{k-2}\}$. So, this establishes $\beta_{[\pm 1]}(HF(k)) = 2$, $k > 1$.

3.6 Discussion and open problems

From the above discussion, it is clear that discrepancy of a set system $\mathcal{A}$ can be arbitrarily large as compared to $\beta_{[\pm 1]}(\mathcal{A})$. On the other extreme, we know that discrepancy of a family of 2-sized subsets $\mathcal{A}$ of $[n]$ cannot exceed 2, whereas $\beta_{[\pm 1]}(\mathcal{A})$ can be as large as $\log n$. Thus, there exists families $\mathcal{A}$ and $\mathcal{G}$ where $\beta_{[\pm 1]}(\mathcal{A})$ and $disc(\mathcal{G})$ are constants whereas $disc(\mathcal{A})$ and $\beta_{[\pm 1]}(\mathcal{G})$ are arbitrarily large. However, this does not rule out a possible relationship between these two parameters and other hypergraph parameters. One possibility of making progress in this direction is obtaining tight upper and lower bounds for $\beta_{[\pm 1]}(\mathcal{A})$. Recall that the discrepancy of a family $\mathcal{A}$ is the minimum $i \in \mathbb{N}$ such that $\beta_{[\pm i]}(\mathcal{A}) \leq 1$. Below, we demonstrate the usage of such tight bounds where $\mathcal{A} = 2^{[n]}$ and n is a power of 2. From Theorem 3.1, we have, $\frac{n}{2} \geq \beta_{[\pm 1]}(n) \geq 2\beta_{[\pm 2]}(n) \geq \cdots \geq 2^j \beta_{[\pm 2^j]}(n)$. So, when $j = \log(\frac{n}{2})$, we get, $\beta_{[\pm 2^j]}(n) \leq 1$. This gives a known trivial upper bound for $disc(\mathcal{A})$.

As mentioned in the introduction, $\beta_{[\pm 1]}(E)$ is $\lceil \log \chi(G) \rceil$ for a graph $G(V, E)$. We know that it is impossible to approximate the chromatic number of graphs on n vertices within a factor of $n^{1-\epsilon}$ for any fixed $\epsilon > 0$, unless $NP \subseteq ZPP$ (see Feige and Killian [27]). Therefore, it is not difficult to see that under the assumption $NP \not\subseteq ZPP$, no polynomial time algorithm can approximate $\beta_{[\pm 1]}(E)$ for an n-vertex graph $G(V, E)$ within an additive approximation factor of $(1 - \epsilon) \log n - 1$, for any fixed $\epsilon > 0$.

Chapter 4

Induced bisecting families for sets of bi-colorings

4.1 Introduction

Two n-dimensional vectors A and B, $A, B \in \mathbb{R}^n$, are said to be *trivially orthogonal* if in every coordinate $i \in [n]$, at least one of $A(i)$ or $B(i)$ is zero. The vectors A and B are *non-trivially orthogonal* if they are orthogonal, but not trivially orthogonal. For instance, the rows of any Hardamard matrix are pairwise non-trivially orthogonal. Consider the following problem: "Given the n-dimensional Hamming cube $\{0, 1\}^n$, what is the minimum cardinality of a subset $\mathcal{V}$ of n-dimensional $\{-1, 0, 1\}$ vectors, each containing exactly d non-zero entries, such that every point $A \in \{0, 1\}^n$ in the Hamming cube has some $V \in \mathcal{V}$ which is non-trivially orthogonal to A?". It is not hard to see that the all-zero vector and the unit vectors $\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$ can never have any non-trivially orthogonal vector in $\{-1, 0, 1\}^n$. Additionally, the all-ones vector $(1, \ldots, 1)$ cannot be non-trivially orthogonal to any vector in $\{-1, 0, 1\}^n$ consisting of exactly d non-zero entries, when d is odd. We call the all zero vector $(0, \ldots, 0)$, the n unit vectors $(1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ (and additionally, $(1, \ldots, 1)$ when d is odd) as *trivial*. Since no n-dimensional $\{-1, 0, 1\}$ vector with exactly one non-zero entry is non-trivially orthogonal to any non-trivial point of the Hamming cube, we assume that $d \geq 2$ in the rest of the chapter.

Definition 4.1 *Let $2 \leq d \leq n$, where d and n are integers. We define $\beta^d(n)$ as the minimum cardinality of a subset $\mathcal{V}$ of n-dimensional $\{-1, 0, 1\}$ vectors, each containing exactly d non-zero entries, such that every non-trivial point in the Hamming cube $\{0, 1\}^n$ has a non-trivially orthogonal vector $V \in \mathcal{V}$.*

In this chapter, we study the problem of estimation of bounds for $\beta^d(n)$.

We now define a general version of the aforementioned problem in terms of bicolorings of a hypergraph. Let G be a hypergraph on the vertex set $[n]$. Corresponding to the trivial vectors/points of the Hamming cube, the singleton sets and the empty set (and additionally, the set $[n]$ when d is odd) are the *trivial hyperedges* or *trivial subsets* of $[n]$. Let Y^S denote a ± 1 bicoloring of vertices of $S \subseteq [n]$, i.e. $Y^S : S \rightarrow \{+1, -1\}$, for some $S \subseteq [n]$. We abuse the notation to denote the subset of vertices colored with +1 (-1) with respect to bicoloring Y^S as $Y^S(+1)$ (resp., $Y^S(-1)$).

Definition 4.2 *Given a hypergraph G, a hyperedge $A \in E(G)$ is said to be* ***induced bisected*** *by a bicoloring Y^S of a subset $S \subseteq V(G)$, if $|A \cap Y^S(+1)| = |A \cap Y^S(-1)| \neq 0$. A set $\mathcal{Y} = \{Y^{S_1}, \ldots, Y^{S_t}\}$ of t bicolorings is called an* ***induced bisecting family of order*** *d for G if*

1. *each $S_i \subseteq [n]$ has exactly d vertices, $1 \leq i \leq t$, $2 \leq d \leq n$, and*
2. *every non-trivial hyperedge $A \in E(G)$ is induced bisected by at least one Y^{S_i}, $1 \leq i \leq t$.*

Let $\beta^d(G)$ denote the minimum cardinality of an induced bisecting family of order d for hypergraph G.

From Definitions 4.1 and 4.1, it is clear that the maximum of $\beta^d(G)$ over all hypergraphs G on $[n]$ is $\beta^d(n)$.

Example 4.3 *Let $\mathcal{H}$ be the hypergraph with all the $2^n - n - 1$ non-trivial subsets of $[n]$ as hyperedges and let $d = 2$. For any $S \in \binom{[n]}{2}$, let Y^S color one point in S with color +1 and the other with -1. Observe that $\mathcal{Y} = \{Y^S | S \in \binom{[n]}{2}\}$ forms an induced bisecting family of order 2 for $\mathcal{H}$. $\beta^2(\mathcal{H}) \leq \binom{n}{2}$. Moreover, this upper bound is also tight: if $Y^{\{a,b\}} \notin \mathcal{Y}$, $\{a, b\} \in \binom{[n]}{2}$, then the hyperedge $\{a, b\} \in \mathcal{H}$ cannot be induced bisected.*

4.1.1 Relations to bisecting families

The problem addressed in this chapter can be viewed as a generalization of the problem of *bisecting families*. Let $n \in \mathbb{N}$ and let $\mathcal{A}$ be a family of subsets of $[n]$. Recall that a family $\mathcal{B}$ of subsets of $[n]$ is called a *bisecting family* for $\mathcal{A}$, if for each $A \in \mathcal{A}$, there exists a $B \in \mathcal{B}$ such that $|A \cap B| \in \{\lceil \frac{|A|}{2} \rceil, \lfloor \frac{|A|}{2} \rfloor\}$. In the bicoloring terminology, letting $S = [n]$, $Y^S(+1) = B$, $Y^S(-1) = [n] \setminus B$, the bisecting family $\mathcal{B}$ maps to a collection $\mathcal{Y}$ of bicolorings such that for each $A \in \mathcal{A}$, there exists a bicoloring $Y \in \mathcal{Y}$ such that $|A \cap Y(+1)| - |A \cap Y(-1)| \in \{-1, 0, 1\}$. $\beta_{[\pm 1]}(n)$ is the minimum cardinality of a bisecting family for the family consisting of all the non-empty subsets of $[n]$. The upper bound constructions given in Chapter 3.2.1 establishing $\beta_{[\pm 1]}(n) = \lceil \frac{n}{2} \rceil$ come from bicolorings that color all the n points, half of them of each of the two colors. The question that naturally arises is: "are there *structurally different bicolorings* that achieve similar bounds?". One notion of such structurally different bicolorings is when the colorings have small support, i.e., only d out of the n points can be colored at a time. This leads us to the question of induced bisection. As it turns out, small support translates to large cardinality of the family of bicolorings, whereas when $d = n - 1$, only n bicolorings suffice.

Let $\mathcal{A}_e$ denote the family of non-trivial even subsets of $[n]$. Note that when $d = n$, any induced bisecting family of order d for $\mathcal{A}_e$ is a bisecting family for the family consisting of all the non-empty subsets of $[n]$. In other words, $\beta^n(\mathcal{A}_e) = \beta_{[\pm 1]}(n)$. However, when $d = n$, i.e. $S = [n]$, no odd subset of $[n]$ can be induced bisected: this follows from the fact that for any odd subset A, $|A \cap Y^S(+1)| - |A \cap Y^S(-1)|$ is odd. When the colorings have small support, the odd subsets of $[n]$ can be induced bisected.

Main result

In this chapter, we establish the following theorem.

Theorem 4.4 *Let $2 \leq d \leq n$, where d and n are integers. Then, $\frac{2n(n-1)}{d^2} \leq \beta^d(n) \leq \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$. Moreover, $\beta^d(n) \geq n-1$, when d is odd.*

This establishes asymptotically tight bounds on $\beta^d(n)$ for all values of n, when d is odd. Moreover, the bound is asymptotically tight when $d \in O(\sqrt{n})$, even if d is even. However, when $d \in \Omega(n^{0.5+\epsilon})$ and d is even, the above lower bound may not be asymptotically tight, for any ϵ, $0 < \epsilon \leq 0.5$.

4.2 Some quick observations

It is not hard to see that $\beta^d(n)$ increases monotonically with increasing n. For any two families $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_1 \subseteq \mathcal{F}_2$, $\beta^d(\mathcal{F}_1) \leq \beta^d(\mathcal{F}_2)$. So, in order to obtain bounds for $\beta^d(n)$, it suffices to compute bounds for the family $\mathcal{F}$ consisting of all the non-trivial subsets of $[n]$.

Firstly, we invoke a coloring procedure similar to the one used in proof of Lemma 3.10 that computes a bisecting family for the $2^d - 1$ non-empty subsets of S, given any $S \subseteq [n]$, $|S| = d \geq 2$.

Algorithm 1: Computing a bisecting family of size $\frac{d}{2}$ for the family $\mathcal{F} = \{e | e \subseteq S\}$.

Data: a subset $S \subseteq [n]$, $|S| = d \geq 2$

Result: a set of bicolorings of $\{Y^{S_1}, \ldots, Y^{S_{\frac{d}{2}}}\}$

for $i = 1$ *to* $\frac{d}{2}$ **do**

$S_i = S$;

$Y^{S_i}(+1) = \{i, \ldots, i + \frac{d}{2} - 1\}$;

$Y^{S_i}(-1) = S \setminus Y^{S_i}(+1)$;

On a set $S \subseteq [n]$, $|S| = d \geq 2$, as input, Algorithm 1 outputs a set of bicolorings of $\mathcal{Y} = \{Y^{S_1}, \ldots, Y^{S_{\frac{d}{2}}}\}$. Proceeding along lines outlined in the proof of Lemma 3.10, it is not hard to show that $\{Y^{S_1}(+1), \ldots, Y^{S_{\frac{d}{2}}}(+1)\}$ forms a bisecting family for $2^d - 1$ non-empty subsets of S. In other words, for every even hyperedge $A_e \subseteq S$, there exists a bicoloring $Y^S \in \mathcal{Y}$ such that $|A_e \cap Y^S(+1)| = |A_e \cap Y^S(-1)| \neq 0$. This observation implies that for any hyperedge $B \subseteq [n]$, if $|B \cap S|$ is not zero and is even, then B is induced bisected by some $Y^S \in \mathcal{Y}$. So, we have the following lemma.

Lemma 4.5 *Let $\mathcal{H}$ be any hypergraph. If $\mathcal{F}'$ consists of d-sized subsets of $[n]$ such that for every $B \in E(\mathcal{H})$, there exists an $A' \in \mathcal{F}'$ such that $|B \cap A'|$ is not zero and is even, then we can construct an induced bisecting family of cardinality $|\mathcal{F}'|\frac{d}{2}$ for $\mathcal{H}$.*

When restricting the problem to the domain of even hyperedges of $[n]$, we can use the properties of a 2-$(n, d, 1)$ design and show that any 2-$(n, d, 1)$ design indeed forms a family $\mathcal{F}'$ fulfilling the requirements of Lemma 4.5. Recall that t-(n, k, λ) design is an incidence structure on n vertices, consisting of k-sized blocks where every t-sized subset of vertices is present in exactly λ blocks, where $k, t, \lambda, n \in \mathbb{N}$. For example, a 2-$(n, d, 1)$ design denotes the structure on n vertices consisting of d sized blocks, where every pair of points belong to exactly one block. It is well known that a 2-$(n, d, 1)$ design consists of exactly $\frac{n(n-1)}{d(d-1)}$ blocks.

Proposition 4.6 *Let $\mathcal{H}_e$ denote the hypergraph consisting of all the non-trivial even hyperedges. Then, $\beta^d(\mathcal{H}_e) \leq \frac{n(n-1)}{2(d-1)}$, when a 2-$(n, d, 1)$ design exists.*

Proof. Let $\mathcal{D}$ denote the set of all the d-sized blocks that form the 2-$(n, d, 1)$ design. Note that if every hyperedge in $E(\mathcal{H}_e)$ has an even and non-zero intersection with some $D \in \mathcal{D}$, then using Lemma 4.5, we get the desired upper bound. In the rest of the proof, we show that for every hyperedge $A \in E(\mathcal{H}_e)$, there exists an $D \in \mathcal{D}$ such that $A \cap D \neq \phi$ and $|A \cap D|$ is even. For the sake of contradiction, assume that there exists some $A \in E(\mathcal{H}_e)$ such that for every block $D \in \mathcal{D}$, either $A \cap D$ is empty, or $|A \cap D|$ is odd. Consider the family $\mathcal{D}' = \{A \cap D | D \in \mathcal{D} \text{ and } |A \cap D| \geq 1\}$. Note that every block $D' \in \mathcal{D}'$ has an odd cardinality - otherwise, our assumption is violated. Observe that for every pair $\{a, b\} \in A$, there exists exactly one $D' \in \mathcal{D}'$ such that $\{a, b\} \in D'$; the sizes of all the $D' \in \mathcal{D}'$ may not be the same. So, $\mathcal{D}'$ forms a design with blocks of possibly different sizes on vertex set A. If $\mathcal{D}'_a$ denotes the set of all the blocks in $\mathcal{D}'$ that contain a, the blocks in $\{D' \setminus \{a\} : D' \in \mathcal{D}'_a\}$ partitions the set $A \setminus \{a\}$. Therefore, $\sum_{D' \in \mathcal{D}'_a}(|D'| - 1) = |A| - 1$. This is a contradiction since (i) $|A|$ is even, $|A| - 1$ is odd, and, (ii) each block D' has odd cardinality by assumption. This establishes that there exists at least one block $D \in \mathcal{D}$ such that $|A \cap D|$ is even. $\square$

The necessary conditions for existence of a 2-$(n, d, 1)$ design are (i) $d - 1$ divides $n - 1$, and (ii) $\binom{d}{2}$ divides $\binom{n}{2}$. Richard Wilson [74, 75, 76] proved that for sufficiently large n, the necessary conditions are also sufficient. Note that even if the necessary conditions are violated, there exist *covering designs* of size $\frac{n(n-1)}{d(d-1)}(1 + o(1))$ provided $d = o(n)$ [61] and $\frac{n(n-1)}{d(d-1)} \log n$ for arbitrary d [70, 44].

4.3 Lower Bounds

Let $\mathcal{H}$ denote the hypergraph consisting of all the non-trivial subsets of $[n]$. Let the set $\mathcal{Y} = \{Y^{S_1}, \ldots, Y^{S_t}\}$ of bicolorings be any optimal induced bisecting family of order d for $\mathcal{H}$, where $t \in \mathbb{N}$.

Considering only the two sized subsets of $[n]$, we note that every two element hyperedge $\{a, b\}$, a,$b \in [n]$, must lie in some S_i, $S_i \in \{S_1, \ldots, S_t\}$; otherwise, no bicoloring in $\mathcal{Y}$ can induced bisect $\{a, b\}$. So, it follows that $\sum_{Y^S \in \mathcal{Y}} \binom{d}{2} \geq \binom{n}{2}$, i.e., $\beta^d(n) \geq \frac{n(n-1)}{d(d-1)}$. A constant factor improvement in the lower bound can be obtained by the following observation: the maximum number of two element subsets $\{a, b\}$ that can be induced bisected by any $Y^S \in \mathcal{Y}$, $|S| = d$, is $\frac{d^2}{4}$. So, we have the following proposition.

Proposition 4.7 $\beta^d(n) \geq \frac{2n(n-1)}{d^2}$.

Observe that when d is large, say $d \in \Omega(n^{0.5+\epsilon})$, where $0 < \epsilon \leq 0.5$, Proposition 4.7

only yields a sublinear lower bound. When d is odd, we can prove a general lower bound of $n-1$ on $\beta^d(n)$ using the following version of Cayley-Bacharach theorem by Riehl and Graham [60] on the maximum number of common zeros between n quadratics and any polynomial P of smaller degree.

Theorem 4.8 *[60] Given the n quadratics in n variables $x_1(x_1-1),\ldots,x_n(x_n-1)$ with 2^n common zeros, the maximum number of those common zeros a polynomial P of degree k can go through without going through them all is 2^n-2^{n-k}.*

Lemma 4.9 *$\beta^d(n)\geq n-1$, when d is odd.*

Proof. Let $\mathcal{B}$ be a induced bisecting family of minimum cardinality for all the non-trivial subsets $A\subseteq[n]$. Let Y_B denote the n-dimensional vector representing the bicoloring $B\in\mathcal{B}$, i.e. $Y_B\in\{-1,0,1\}^n$ and Y_B contains exactly d nonzero entries. Consider the polynomials $M(X)$, $N(X)$, and $P(X)$, $X\in\{0,1\}^n$.

$$M(X=(x_1,\ldots,x_n))=\prod_{B\in\mathcal{B}}\langle Y_B,X\rangle\,. \tag{4.1}$$

$$N(X=(x_1,\ldots,x_n))=(\sum_{i=1}^{n}x_i)-1. \tag{4.2}$$

$$P(X)=M(X)N(X). \tag{4.3}$$

Let X_A denote the 0–1 n-dimensional incidence vector corresponding to $A\subseteq[n]$. Note that $M(X_A)$ vanishes for each $A\subseteq[n]$ except (i) the all 1's vector, $(1,\ldots,1)\}$, since d is odd, and (ii) possibly the singleton sets. Since $N(X_A)$ vanishes for all singleton sets, $P(X_A)$ vanishes on all subsets $A\subseteq[n]$ except for the set $[n]$ (corresponding to the all 1's vector). Since the degree of P is $|\mathcal{B}|+1$ and P in non-zero only at $X_A=(1,\ldots,1)\}$, using Theorem 4.8, we have $|\mathcal{B}|\geq n-1$. □

However, when d is even, the above lower bounding technique does not work since the polynomial M may vanish at every point of the Hamming cube $\{0,1\}^n$. In this case, we can obtain a lower bound of $\Omega(\sqrt{d})$ (we focus on only a fixed d sized block that has $2^d - 1$ non-empty subsets out of which the maximum number of subsets that can be induced-bisected by a single bicoloring is $(\binom{d}{d/2})^2$; so minimum $\Omega(2^d/(\binom{d}{d/2})^2) = \Omega(\sqrt{d})$ such bicolorings are needed).

4.4 Induced bisecting families when n is $d+1$

In what follows, we consider the hypergraph $\mathcal{H}$ consisting of all the non-trivial hyperedges of $[n]$, where $n = d+1$ and demonstrate a construction of an induced bisecting family of order d of cardinality $d+1$.

Theorem 4.10 *Let d be an integer greater than 1. Then, $d \leq \beta^d(d+1) \leq d+1$. Moreover, $\beta^d(d+1) = d+1$, when d is even.*

Proof. We consider the cases when d is even and d is odd separately. We start our analysis with the case when d is even.

Case 1: d is even

Let $v_1, \dots, v_{d+1}$ denote the $d+1$ vertices. Consider a circular clockwise arrangement of $d+1$ slots, namely $P_1, \dots, P_{d+1}$ in that order. The slots P_1 to $P_{\frac{d}{2}}$ are colored with +1, slots $P_{\frac{d}{2}+2}$ to P_{d+1} are colored with -1, and only slot $P_{\frac{d}{2}+1}$ remains uncolored. Each slot can contain exactly one vertex and each vertex takes the color of the slot it resides in. As for the initial configuration, let $v_i \in P_i$, for $1 \leq i \leq d+1$. This configuration gives the coloring Y_1, where (i) $Y_1(+1) = \{v_1, \dots, v_{\frac{d}{2}}\}$, (ii) $Y_1(-1) = \{v_{\frac{d}{2}+2}, \dots, v_{d+1}\}$, and, (iii) the vertex $v_{\frac{d}{2}+1}$ remains uncolored. We obtain the second coloring Y_2 from Y_1 by one clockwise rotation of the vertices in the circular arrangement. Therefore, we have, $Y_2(+1) = \{v_{d+1}, v_1, \dots, v_{\frac{d}{2}-1}\}$, $Y_2(-1) = \{v_{\frac{d}{2}+1}, \dots, v_d\}$; the vertex $v_{\frac{d}{2}}$ remains uncolored. See Figure 4.1 for an illustration. Similarly, repeating the process d times, we obtain the set $\mathcal{Y} = \{Y_1, \dots, Y_{d+1}\}$ of bicolorings. We have the following observations.

Observation 4.11 *If $\mathcal{Y}$ induced bisects every non-trivial odd subset of $[d+1]$, then $\mathcal{Y}$ induced bisects every non-trivial even subset of $[d+1]$ as well.*

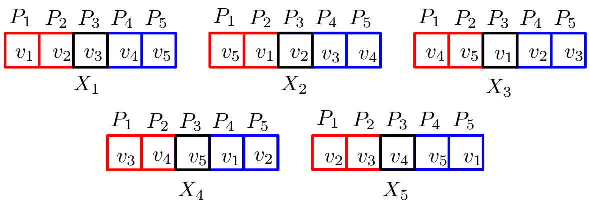

Figure 4.1: Vertices in (i) P_1 and P_2 are colored with +1, (ii) P_4 and P_5 are colored with -1; the vertex in P_3 remains uncolored. $\mathcal{Y} = \{Y_1, \ldots, Y_5\}$ is an induced bisecting family when $n = d + 1 = 5$.

To prove the observation, consider an even hyperedge $A_e \subset [d+1]$, and let $Y \in \mathcal{Y}$ be the bicoloring that induced bisects the odd hyperedge $\bar{A}_e = [d+1] \setminus A_e$. Note that one vertex in $\bar{A}_e$ remains uncolored under Y. Otherwise, $\bar{A}_e$ cannot get induced bisected under Y. Since $|Y(+1)| = \frac{d}{2}$ and $|\bar{A}_e \cap Y(+1)| = \frac{|\bar{A}_e|-1}{2}$, it follows that $|A_e \cap Y(+1)| = |Y(+1) \setminus (\bar{A}_e \cap Y(+1))| = \frac{d}{2} - \frac{|\bar{A}_e|-1}{2}$. Similarly, $|A_e \cap Y(-1)| = \frac{d}{2} - \frac{|\bar{A}_e|-1}{2}$. So, A_e is induced bisected under Y. This completes the proof of Observation 4.11.

Therefore, it suffices to prove that $\mathcal{Y}$ induced bisects every non-trivial odd subset of $[d+1]$. For the sake of contradiction, assume that A is an odd hyperedge not induced bisected by $\mathcal{Y}$. Let $c_i = |A \cap Y_{i+1}(+1)| - |A \cap Y_{i+1}(-1)|$, for $0 \leq i \leq d$. All additions/subtractions in the subscript of c are modulo $d+1$. Our assumption implies that $c_i \neq 0$ for all $0 \leq i \leq d$.

Observation 4.12 $|c_i - c_{i+1}| \leq 2$, *for* $0 \leq i \leq d$. *Furthermore, if* $c_i > c_{i+1}$ *and* c_i *is odd, then* $c_i - c_{i+1} = 1$.

The first part of Observation 4.12 follows from the construction and we omit the details for brevity. Note that when c_i is odd, the element in $P_{\frac{d}{2}+1}$ cannot belong to the odd hyperedge A. This takes care of the second part of Observation 4.12.

Observe that a bicoloring $Y_j \in \mathcal{Y}$, $1 \leq j \leq d+1$, induced bisects the odd hyperedge A if and only if c_j is 0. We know that bicoloring Y_2 (Y_{i+1}) is obtained from Y_1 (Y_i, respectively) by one clockwise rotation of vertices in the circular arrangement. Thus, during the construction of bicolorings Y_1 through Y_{d+1}, we perform a full rotation of the vertices with respect to their starting arrangement in Y_1. So, it follows that there exist i and j such that c_i is positive and c_{i+j} is negative. Combined with the second part of

Observation 4.12, this implies the existence of an index p such that $c_p = 0$. This is a contradiction to the assumption that A is not induced bisected by $\mathcal{Y}$. Therefore, every odd subset of $[d+1]$ is induced bisected by $\mathcal{Y}$, and using Observation 4.11, the upper bound on $\beta^d(d+1)$ follows.

To see that the upper bound is tight, observe that exactly one d-sized hyperedge can get induced bisected under a single bicoloring - the hyperedge missing the uncolored vertex. This completes the proof of Theorem 4.10 for even values of d.

Case 2: d is odd

The fact that $\beta^d(d+1) \geq d$ for odd values of d follows directly from Lemma 4.9. Recall that along with the empty set and the singleton sets, the set $[d+1]$ becomes trivial when d is odd. In the initial configuration, let $v_i \in P_i$, for $1 \leq i \leq d+1$. This configuration gives the coloring Y_1, where (i) $Y_1(+1) = \{v_1, \ldots, v_{\frac{d+1}{2}-1}\}$, (ii) $Y_1(-1) = \{v_{\frac{d+1}{2}+1}, \ldots, v_{d+1}\}$, and, (iii) the vertex $v_{\frac{d+1}{2}+1}$ remains uncolored. We obtain the second coloring Y_2 from Y_1 by one clockwise rotation of the vertices in the circular arrangement. Therefore, we have, $Y_2(+1) = \{v_{d+1}, v_1, \ldots, v_{\frac{d+1}{2}-2}\}$, $Y_2(-1) = \{v_{\frac{d+1}{2}}, \ldots, v_d\}$; the vertex $v_{\frac{d+1}{2}-1}$ remains uncolored. Similarly, repeating the rotation d times, we obtain the set $\mathcal{Y} = \{Y_1, \ldots, Y_{d+1}\}$ of bicolorings.

The proof for $\mathcal{Y}$ being an induced bisecting family for any odd hyperedge $A_o \subsetneq [d+1]$ is exactly similar to that given in the proof of Theorem 4.10. So, we consider only the even hyperedges. Let $c_i = |A \cap Y_{i+1}(+1)| - |A \cap Y_{i+1}(-1)|$, for $0 \leq i \leq d$. All additions/subtractions in the subscript of c are modulo $d+1$. For the sake of contradiction, assume that A is an even hyperedge not induced bisected by $\mathcal{Y}$. If we can show that some c_j, $0 \leq j \leq d$, is zero, then we get the desired contradiction.

Observation 4.13 $|c_i - c_{i+1}| \leq 2$, *for* $0 \leq i \leq d$.

The proof of Observation 4.13 follows from the construction. Consider the sequence $(c_i, c_{i+1}, \ldots, c_{i+d+1})$, where $c_i \leq c_j$, $j \in \{i+1, \ldots, i+d+1\}$, and the addition is modulo $d+1$. Since there is a full rotation of the vertex set with respect to the slots, it follows that (i) $c_i \leq 0$, and (ii) there exists another index j such that c_j is positive. From

Observation 4.13, it follows that if none of the c_j, $j \in \{0, \ldots d\}$, is zero, there exists an index p such that $c_p = -1$ and $c_{p+1} = 1$. Note that $c_p = -1$ asserts that $A \cap P_{\frac{d+1}{2}}$ is non-empty. However, under this configuration, c_{p+1} can never become 1. This yields the desired contradiction. □

We have the following corollary which gives an upper bound to the cardinality of an induced bisecting family for arbitrary values of n.

Corollary 4.13.1 *Let $\mathcal{H}$ be any hypergraph on vertex set $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ and let $d \in [n]$. Let $\mathcal{F}$ consist of $(d+1)$-sized subsets of $V(\mathcal{H})$ such that for every $B \in E(\mathcal{H})$, there exists an $A \in \mathcal{F}$ with (i) $|B \cap A| \geq 2$, when d is even; (ii) $2 \leq |B \cap A| \leq d$, when d is odd. Then, we can construct an induced bisecting family of order d of cardinality $|\mathcal{F}|(d+1)$ for $\mathcal{H}$.*

Proof. For any subset $A \in \mathcal{F}$, using the procedure used in the proof of Theorem 4.10, we can obtain an induced bisecting family $\mathcal{Y}_A$ for all the non-trivial subsets of A, where $|\mathcal{Y}_A| = d+1$. When d is even, $\mathcal{Y}_A$ induced bisects all the $2^{d+1} - (d+1) - 1$ non-empty and non-singleton subsets of A; therefore, each $B \in E(\mathcal{H})$ with $|B \cap A| \geq 2$ is induced bisected by $\mathcal{Y}_A$. When d is odd, $\mathcal{Y}_A$ induced bisects all but the empty set, the singleton sets, and A; so, each $B \in E(\mathcal{H})$ with $2 \leq |B \cap A| \leq d$ is induced bisected by $\mathcal{Y}_A$. Repeating the process for each $A \in \mathcal{F}$, we get an induced bisecting family of cardinality $|\mathcal{F}|(d+1)$ for $\mathcal{H}$. □

Theorem 4.10 provides evidence for the following property (which is described in Corollary 4.13.2) of the odd subsets under any circular permutation of odd number of elements which may be of independent interest. For any circular permutation σ of $[n]$, $a, b \in [n]$, let $dist_\sigma(a, b)$ denote the clockwise distance between a and b with respect to σ, which is one more than the number of elements residing between a and b in the permutation σ in the clockwise direction.

Corollary 4.13.2 *Consider any circular permutation σ of $[n]$, where n is odd. For any odd k-sized subset $A \subseteq [n]$, let $(a_0, \ldots, a_{k-1})$ be the ordering of A with respect to σ.*

Then, there exists an index $i \in \{0, \dots, k-1\}$ such that $dist_\sigma(a_i, a_{i+\lfloor \frac{k}{2} \rfloor}) < \frac{n}{2}$ and $dist_\sigma(a_{i+\lfloor \frac{k}{2} \rfloor + 1}, a_i) < \frac{n}{2}$, where summation in the subscript of a is modulo k.

Proof. Consider a circular clockwise arrangement of n slots, namely $P_1, \dots, P_n$ in that order. Put vertex $\sigma(i)$ in P_i. Now, following the procedure outlined in the proof of Theorem 4.10, obtain a bicoloring that bisects A. Pick the uncolored vertex residing in slot $P_{\lceil \frac{n}{2} \rceil}$ with respect to the bicoloring Y. Observe that this vertex satisfies the desired property. □

As noted in the introduction, when $d = n$, the odd hyperedges of a hypergraph $\mathcal{H}$ can never get bisected (for example, consider any $(n-1)$-sized hyperedge). Let $\mathcal{H}_e$ and $\mathcal{H}_o$ denote the hypergraphs consisting of the nontrivial even subsets and the nontrivial odd subsets of $[n]$, respectively. In Section 4.1, we observed that $\beta^n(\mathcal{H}_e) = \lceil \frac{n}{2} \rceil$, So, a natural question in this direction is - when the hypergraph under consideration consists of only the odd hyperedges, what is the minimum cardinality of an induced bisecting family, provided we can choose arbitrary value of d? In what follows, we address this problem and we have the following theorem.

Theorem 4.14 *Let $\mathcal{H}_o$ denote the hypergraph consisting of the nontrivial odd subsets of $[d+2]$. $\beta^d(\mathcal{H}_o) = \frac{d}{2} + 1$, when d is an even integer.*

Proof. In what follows, we give an explicit construction of $\frac{d}{2}+1$ sized induced bisecting family of order d for $\mathcal{H}_o$ to establish the upper bound. Let $v_1, \dots, v_{d+2}$ denote the $d+2$ vertices. As in the previous construction, consider a circular arrangement of n slots, namely $P_1, \dots, P_n$. The slots P_1 to $P_{\frac{d}{2}}$ are colored with +1, slots $P_{\frac{d}{2}+2}$ to P_{d+1} are colored with -1, and slots $P_{\frac{d}{2}+1}$ and P_{d+2} are uncolored. Each slot can contain exactly one vertex and each vertex is colored according to the color of the slot it resides in. For coloring Y_1, we place vertex v_i in P_i, $1 \le i \le n$. So, $Y_1(+1) = \{v_1, \dots, v_{\frac{d}{2}}\}$, $Y_1(-1) = \{v_{\frac{d}{2}+2}, \dots, v_{d+1}\}$. The vertices $v_{\frac{d}{2}+1}$ and v_{d+2} remain uncolored. We obtain the second coloring Y_2 from Y_1 by one clockwise rotation of the vertices in the circular arrangement. So, in Y_2, P_i contains the element v_{i-1}, $2 \le i \le d+2$, P_1 contains

v_{d+2}. Similarly, repeating the process $\frac{d}{2}$ times, we obtain the set $\mathcal{Y} = \{Y_1, \ldots, Y_{\frac{d}{2}+1}\}$ of bicolorings.

The proof that $\mathcal{Y}$ is indeed an induced bisecting family for $\mathcal{H}_o$ is similar to the previous case; we omit the details for brevity. The lower bound $\beta^d(\mathcal{H}_o) \geq \frac{n}{2}$, $d = n-2$, can be obtained considering only the $(n-1)$-sized sets. In order to induced bisect an $(n-1)$-sized set A using any coloring Y^S of a set S, $|S| = n-2$, observe that (i) $Y^S(+1)$ must be equal to $Y^S(-1)$, and both are of size $\frac{n}{2} - 1$, (ii) $S \subset A$, and (iii) A must miss a vertex in $[n] \setminus S$. So, any fixed $(n-2)$-sized set can induced bisect at most 2 $(n-1)$-sized sets. Since there are n such sets, it follows that we need at least $\frac{n}{2}$ bicoloring of $(n-2)$-sized subsets to cover all the $(n-1)$-sized sets. $\square$

4.5 Upper bounds for $\beta^d(n)$ and proof of Theorem 4.4

From Proposition 4.7, we know that $\beta^d(n) \geq \frac{2n(n-1)}{d^2}$. In this section, we prove an upper bound of $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$ for $\beta^d(n)$.

4.5.1 Deterministic construction of induced bisecting families

Lemma 4.15 $\beta^d(n) \leq \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$.

Proof idea. For simplicity, assume that d is even and d divides n. The idea for the proof is to first partition n into $\frac{d}{2}$-sized parts: let $\mathcal{P} = \{P_1, \ldots, P_{\frac{2n}{d}}\}$ denote this partition. For every distinct $P_i, P_j \in \mathcal{P}, i < j$, construct distinct bicolorings where points in P_i (P_j) are colored +1 (respectively, -1). This results in $\binom{\frac{2n}{d}}{2}$ bicolorings. All subsets of $[n]$ that intersect with every part of $\mathcal{P}$ in at most one point are induced bisected by the above set of bicolorings. So, the remaining subsets intersect with some $\frac{d}{2}$ sized part of $\mathcal{P}$ in at least two points. We perform few extra bicolorings to induced bisect subsets.

Before proceeding to the proof of the above lemma, we introduce few definitions that simplify the proof considerably. Let d be a positive even integer. Let $\mathcal{S}(n,d) = \{P_1, \ldots, P_{\lceil \frac{2n}{d} \rceil}\}$ denote a partition of $[n]$, where each $P \in \mathcal{S}(n,d) \setminus \{P_{\lceil \frac{2n}{d} \rceil}\}$ is of cardinality exactly $\frac{d}{2}$, and $|P_{\lceil \frac{2n}{d} \rceil}| \leq \frac{d}{2}$. Let $P^1_{\lceil \frac{2n}{d} \rceil} = P_{\lceil \frac{2n}{d} \rceil} \cup Q_1$, $P^2_{\lceil \frac{2n}{d} \rceil} = P_{\lceil \frac{2n}{d} \rceil} \cup Q_2$, where Q_i denotes a fixed $(\frac{d}{2} - |P_{\lceil \frac{2n}{d} \rceil}|)$-sized subset of P_i. For an even d, we define $\mathcal{P}(n, d)$, $\mathcal{D}(n, d)$ and $\mathcal{B}(n, d)$ as follows.

Definition of $\mathcal{P}(n, d)$

$$\mathcal{P}(n,d) = \begin{cases} \mathcal{S}(n,d), \text{ if } \frac{d}{2} \text{ divides } n \\ \mathcal{S}(n,d) \setminus \{P_{\lceil \frac{2n}{d} \rceil}\} \cup \{P^1_{\lceil \frac{2n}{d} \rceil}, P^2_{\lceil \frac{2n}{d} \rceil}\}, \text{ otherwise.} \end{cases} \tag{4.4}$$

Definition of $\mathcal{B}(n, d)$

$\frac{d}{2}$ **divides** n **:** For each $i, j \in \left[\frac{2n}{d}\right]$, $i < j$, let $B_{i,j} : P_i \cup P_j \to \{+1, -1\}$ denote a bicoloring, where

$$B_{i,j}(x) = \begin{cases} +1, \text{ if } x \in P_i \\ -1, \text{ if } x \in P_j. \end{cases}$$

Let $\mathcal{B}(n, d) = \{B_{i,j} | i, j \in \left[\frac{2n}{d}\right], i < j\}$ denote this set of bicolorings.

$\frac{d}{2}$ **does not divide** n **:** For each $i, j \in \left[\lceil \frac{2n}{d} \rceil - 1\right]$, $i < j$, let $B_{i,j} : P_i \cup P_j \to \{+1, -1\}$ denote a bicoloring, where

$$B_{i,j}(x) = \begin{cases} +1, \text{ if } x \in P_i \\ -1, \text{ if } x \in P_j. \end{cases}$$

Let $B_{1,\lceil \frac{2n}{d} \rceil} : P_1 \cup P^2_{\lceil \frac{2n}{d} \rceil} \to \{-1, 1\}$ and $B_{i,\lceil \frac{2n}{d} \rceil} : P_i \cup P^1_{\lceil \frac{2n}{d} \rceil} \to \{-1, 1\}$, for $2 \leq i \leq \lceil \frac{2n}{d} \rceil - 1$.

$$B_{1,\lceil \frac{2n}{d} \rceil}(x) = \begin{cases} +1, \text{ if } x \in P_1 \\ -1, \text{ if } x \in P^2_{\lceil \frac{2n}{d} \rceil} \end{cases}$$

.

$$B_{i,\lceil\frac{2n}{d}\rceil}(x) = \begin{cases} +1, \text{ if } x \in P_i \\ -1, \text{ if } x \in P^1_{\lceil\frac{2n}{d}\rceil} \end{cases}, \text{ for } 2 \le i \le \left\lceil \frac{2n}{d} \right\rceil - 1.$$

Let $\mathcal{B}(n,d) = \{B_{i,j} | i,j \in \left[\lceil\frac{2n}{d}\rceil\right], i < j\}$ denote this set of bicolorings.

Definition of $\mathcal{D}(n,d)$

$\mathcal{D}(n,d) = \{D_k | D_k = P_{2k-1} \cup P_{2k}, k \in \left[\lceil\frac{n}{d}\rceil - 1\right]\} \cup \{D_{\lceil\frac{n}{d}\rceil}\}$, where

$$D_{\lceil\frac{n}{d}\rceil} = \begin{cases} P_{\frac{2n}{d}-1} \cup P_{\frac{2n}{d}}, \text{ if } \frac{d}{2} \text{ divides } n \\ P_1 \cup P^2_{\lceil\frac{2n}{d}\rceil}, \text{ if } \frac{d}{2} \text{ does not divide } n \text{ and } \lceil\frac{2n}{d}\rceil \text{ is odd} \\ P_{\lceil\frac{2n}{d}\rceil-1} \cup P^2_{\lceil\frac{2n}{d}\rceil}, \text{ if } \frac{d}{2} \text{ does not divide } n \text{ and } \lceil\frac{2n}{d}\rceil \text{ is even.} \end{cases} \tag{4.5}$$

Proof. If $d = n-1$, the statement of the lemma follows directly from Theorem 4.10. So, we assume that $d < n-1$ in the rest of the proof. We prove this lemma considering the exhaustive cases based on whether d is even or odd, separately.

Case 1. d is even

Let $\mathcal{P} = \mathcal{P}(n,d)$, $\mathcal{B} = \mathcal{B}(n,d)$ and $\mathcal{D} = \mathcal{D}(n,d)$.

Observation 4.16 *For any $C \subseteq [n]$, $|C| \ge 2$, if $|C \cap P| \le 1$, for all $P \in \mathcal{P}$, then C is induced bisected by at least one $B \in \mathcal{B}$.*

For any $C \subseteq [n]$, $|C| \ge 2$, it follows from the premise that there exist $P_i, P_j \in \mathcal{P}$, $i < j$, such that $|C \cap P_i| = |C \cap P_j| = 1$. C is induced bisected by the bicoloring $B_{i,j}$, thus completing the proof of Observation 4.16.

Let $\mathcal{C}$ denote the family of all the subsets of $[n]$ that are not induced bisected by any $B \in \mathcal{B}$. Rephrasing Observation 4.16, for each $C \in \mathcal{C}$, there exists a $P \in \mathcal{P}$ (and thus, a $D \in \mathcal{D}$) such that $|C \cap P| \ge 2$ (respectively, $|C \cap D| \ge 2$). Let $\mathcal{D}' = \{D \cup \{j\} | j \in [n] \setminus D, D \in \mathcal{D}\}$. Recall that $|D| = d$, where d is an even integer less than $n-1$. So, each $D' \in \mathcal{D}'$ is a $(d+1)$-sized set. Using Corollary 4.13.1, every $C \in \mathcal{C}$ can be induced bisected using $|\mathcal{D}|(d+1)$ bicolorings. Therefore, we have, $\beta^d(n) \leq |\mathcal{B}| + |\mathcal{D}|(d+1) = \binom{\lceil \frac{2n}{d} \rceil}{2} + \lceil \frac{n}{d} \rceil (d+1)$, when d is even.

Case 2. d **is odd**

Let $\mathcal{P} = \mathcal{P}(n-1, d-1)$, $\mathcal{B} = \mathcal{B}(n-1, d-1)$ and $\mathcal{D} = \mathcal{D}(n-1, d-1)$. Since $d-1$ is even, $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{D}$ are well defined. We extend the domain of each $B \in \mathcal{B}$ to $domain(B) \cup \{n\}$, and assign a +1 color to n in each B. Now, each $B \in \mathcal{B}$ colors exactly d elements of $[n]$.

Observation 4.17 *For any $C \subseteq [n]$ with $|C| \geq 2$, if $n \notin C$ and $|C \cap P| \leq 1$ for all $P \in \mathcal{P}$, then C is induced bisected by at least one $B \in \mathcal{B}$.*

The proof of this observation is exactly the same as the proof of Observation 4.16.

Let $\mathcal{C}$ denote the family of all the subsets of $[n]$ that are not induced bisected by any $B \in \mathcal{B}$. For any $D \subseteq [n]$, let $\max(D)$ denote the maximum integer in the set D. Let $\mathcal{D}' = \{D \cup \{n\} \cup \{\max(D)+1\} | D \in \mathcal{D}\}$, where the addition is modulo $n-1$.

Observation 4.18 *Let $\mathcal{D}' = \{D'_1, D'_2, ..., D'_{\lceil \frac{n-1}{d-1} \rceil}\}$ be the family of subsets constructed as above. Then, $|D'_i \cap D'_{i+1}| = 2$, if $1 \leq i \leq \lceil \frac{n-1}{d-1} \rceil - 1$, and $|D'_{\lceil \frac{n-1}{d-1} \rceil} \cap D'_1| \geq 2$.*

Recall that each $D \in \mathcal{D}$ is a $(d-1)$-sized subset of $[n-1]$, where d is an odd integer less than $n-1$. So, each $D' \in \mathcal{D}'$ is a $(d+1)$-sized set. From Observation 4.17, it follows that for each $C \in \mathcal{C}$, there exists at least one $D' \in \mathcal{D}'$ such that $|C \cap D'| \geq 2$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the family of subsets of $[n]$ such that for each $C' \in \mathcal{C}'$, there exists some $D' \in \mathcal{D}'$ such that $2 \leq |C \cap D'| \leq d$. Using Corollary 4.13.1, we can obtain an induced bisecting family for members of $\mathcal{C}'$ of cardinality $|\mathcal{D}|(d+1)$. So, it follows that any $C \in \mathcal{C} \setminus \mathcal{C}'$ must contain one or more elements from $\{D'_1, D'_2, ..., D'_{\lceil \frac{n-1}{d-1} \rceil}\}$ as its subsets.

For any $C \in \mathcal{C} \setminus \mathcal{C}'$, if $D'_i \subseteq C$, then $D'_{i+1} \subseteq C$: otherwise, from Observation 4.18, $2 \leq |C \cap D'_{i+1}| \leq d$ and from definition of $\mathcal{C}'$, $C \in \mathcal{C}'$. So, it follows that $\mathcal{C} \setminus \mathcal{C}' = \{[n]\}$, and $[n]$ is a trivial set when d is odd. Therefore, the cardinality of the induced bisecting family for $[n]$ when d is odd is at most $|\mathcal{B}| + |\mathcal{D}|(d+1) = \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$.

□

4.5.2 Proof of Theorem 4.4

Statement of Theorem 4.4. Let $2 \le d \le n$, where d and n are integers. Then, $\frac{2n(n-1)}{d^2} \le \beta^d(n) \le \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$. Moreover, $\beta^d(n) \ge n-1$, when d is odd.

Proof. Theorem 4.4 follows from Proposition 4.7, Lemma 4.9 and Lemma 4.15. □

Remark 4.19 *By removing some duplicate bicolorings, we can actually improve the upper bound for $\beta^d(n)$ from $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$ to $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil d$.*

Theorem 4.4 asserts an upper bound of $O(n)$ on $\beta^d(n)$ when $d \in \Omega(\sqrt{n})$. Let $k(G)$ denote the minimum cardinality of any hyperedge of the hypergraph G, i.e., $k(G) = \min_{e \in E(G)} |e|$. For any hypergraph G, the upper bound for $\beta^d(G)$ can be improved to $O(n)$ even if $d \in o(\sqrt{n})$ provided $(d-1)k(G) > n-1$ in the following way. Since $(d-1)k(G) > n-1$, every hyperedge is large enough so that the family $\mathcal{D}'$ constructed in all the cases of proof of Lemma 4.15 satisfies the conditions of the family requirements of Corollary 4.13.2. Therefore, the set of bicolorings given by $\mathcal{B} = \mathcal{B}(n,d)$ (or $\mathcal{B}(n-1, d-1)$) can be completely avoided. Thus, we have the following theorem.

Theorem 4.20 *For any hypergraph G, let $k(G) = \min\limits_{e \in E(G)} |e|$. If $(d-1)k(G) > n-1$, then $\beta^d(G) \le \lceil \frac{n-1}{d-1} \rceil (d+1)$.*

Remark 4.21 *The proof of Theorem 4.4 is algorithmic: it yields an induced bisecting family of cardinality at most $\binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$ cardinality with a running time of $O(\frac{n^2}{d^2} + n)$. Observe that the running time of our algorithm is asymptotically equivalent to the cardinality of the family of bicolorings it outputs. Therefore, the asymptotic running time of our algorithm is optimal whenever it outputs an asymptotically optimal solution. Recall that Theorem 4.4 asserts tight bounds for $\beta^d(n)$ except for the case where d is even and $d \in \Omega(n^{0.5+\epsilon})$, for any $0 < \epsilon \le 0.5$.*

We note that if $d = O(1)$, then Theorem 4.4 asserts that $\beta^d(n) = \theta(n^2)$. However, the corresponding coefficients are not the same: the lower bound has the coefficient $\frac{2}{d^2}$ whereas the upper bound has the coefficient $\frac{2}{(d-1)^2}$. It would be interesting to determine the exact coefficient in this case. Moreover, when d is even and $d \in \Omega(n^{0.5+\epsilon})$, for any $0 < \epsilon \leq 0.5$, we have an upper bound of $O(n)$ on $\beta^d(n)$; the lower bound for this case is $o(n)$. We believe that $\beta^d(n)$ is more close to the upper bound and tightening of the bound for $\beta^d(n)$ in this case remains open.

4.6 Exactly bisecting families of intervals

Let $\mathcal{I}$ denote the family consisting of all the valid intervals of $[n]$, where $[n]$ represents points 1 through n in $\mathbb{R}^1$. Let $\mathcal{I}_e$ ($\mathcal{I}_o$) denote the family of all the even (respectively, odd) intervals of $[n]$. Bisecting the even intervals without any restriction on 'the number of colored points' is trivial: consider the bicoloring Y.

$$Y(i) = \begin{cases} +1, & \text{if } i \text{ is an odd integer,} \\ -1, & \text{otherwise.} \end{cases} \tag{4.6}$$

It is easy to see that for every even interval of $I \in \mathcal{I}$, $|I \cap Y(+1)| = |I \cap Y(-1)|$. So $\beta(\mathcal{I}_e) = 1$. Exactly bisecting the family $\mathcal{I}_o$ of odd intervals under induced constraints is slightly tricky. In what follows, we show a simple construction establishing $\beta(\mathcal{I}_o) = 2$.

$\beta^d(\mathcal{I}_o) = 2$, when d is unrestricted

An odd interval $I \in \mathcal{I}_o$ (and corresponding subset) can start at either $4k$, $4k+1$, $4k+2$, $4k+3$ positions, for some integer $k >= 0$ and is of cardinality either $4k'+1$ or $4k'+3$, integer $k' >= 0$. This gives rise to 8 mutually exclusive cases.

Construction 1 *Let S_1 consists of all the points in $[n]$ of the form $4m+2$ and $4m+3$,* *for integer $m >= 0$. The bicoloring $Y^{S_1} : S_1 \rightarrow \{-1, +1\}$ is defined as follows.*

$$Y^{S_1}(i) = \begin{cases} +1, & \text{if } i \in S_1 \text{ is an even integer,} \\ -1, & \text{if } i \in S_1 \text{ is an odd integer.} \end{cases} \tag{4.7}$$

Construction 2 *Let S_2 consists of all the points in $[n]$ of the form $4m$ and $4m + 1$, for integer $m >= 0$. The bicoloring $Y^{S_2} : S_2 \rightarrow \{-1, +1\}$ is defined as follows.*

$$Y^{S_2}(i) = \begin{cases} +1, & \text{if } i \in S_2 \text{ is an even integer,} \\ -1, & \text{if } i \in S_2 \text{ is an odd integer.} \end{cases} \tag{4.8}$$

Claim 1 *The bicolorings Y^{S_1} and Y^{S_2} constructed as per Constructions 1 and 2, respectively, induced bisect every odd interval I of $[n]$.*

Proof. Let k, k' be integers. In order to prove Claim 1, consider the following exhaustive cases based on the starting point of intervals.

1. odd intervals starting at $4k + 1$ - all such intervals are bisected by Y^{S_1} .
2. odd intervals starting at $4k + 3$ - all such intervals are bisected by Y^{S_2}.
3. odd intervals starting at $4k$ with cardinality of the form $4k' + 1$- all such intervals are bisected by Y^{S_1} .
4. odd intervals starting at $4k$ with cardinality of the form $4k' + 3$- all such intervals are bisected by Y^{S_2}.
5. odd intervals starting at $4k + 2$ with cardinality of the form $4k' + 3$- all such intervals are bisected by Y^{S_1} .
6. odd intervals starting at $4k + 2$ with cardinality of the form $4k' + 1$- all such intervals are bisected by Y^{S_2}.

This completes the proof of the claim. □

The above discussion proves that $\beta^d(\mathcal{I}_o) \leq 2$ provided d is unrestricted (or $d = n/2$). We state the result as a lemma below.

Lemma 4.22 *$\beta^d(\mathcal{I}_o) \leq 2$, when d is unrestricted.*

For any family $\mathcal{I}'$ of intervals of some fixed size k, it is not hard to show that one bicoloring may be sufficient to induced bisect every interval of $\mathcal{I}'$. However, with two different sizes of intervals in a family $\mathcal{I}'$, $\beta^d(\mathcal{I}_o)$ may become strictly greater than 1. Consider a permutation $V = (1, 2, 3, 4, 5, 6)$, and $\mathcal{I}_o$ consists of all the odd intervals of cardinality 3 and 5. We consider all possible cardinalities for S_1 and corresponding bicolorings Y^{S_1}, and show that we cannot exactly bisect all the odd intervals using one such set. Note that the cases when $|S_1|$ is 1 and 6 can be safely eliminated from consideration.

- $|S_1| = 2$: Let $S_1 = \{i, j\}$, $i < j$. If $j = i+1$, there is a 3-sized interval containing at most one of them and can never be induced bisected: either $(j, j+1, j+2)$ or $(i-2, i-1, i)$ is a valid interval and is not induced bisected by any bicoloring of S_1. If $j \geq i+2$, either $(j-1, j, j+1)$ or $(i-1, i, i+1)$ is a valid interval and is not induced bisected by any bicoloring of S_1.

- $|S_1| = 3$: Observe that in order to induced bisect the two 5 sized intervals $\{1, 2, 3, 4, 5\}$ and $\{2, 3, 4, 5, 6\}$, we have to color the vertices 1 and 6, and one out of $\{2, 3, 4, 5\}$: if 1 is not colored, $\{2, 3, 4, 5, 6\}$ cannot be induced bisected; if 6 is not colored, $\{1, 2, 3, 4, 5\}$ cannot be induced bisected. However, if both 1 and 6 are colored, then at least one of the 3-sized interval cannot be induced bisected.

- $|S_1| = 4$: Observe that in order to cover the two 5 sized intervals $\{1, 2, 3, 4, 5\}$ and $\{2, 3, 4, 5, 6\}$, either we color both the points 1 and 6, or both 1 and 6 must remain uncolored: if we color vertex 1 and 6 remains uncolored, the hyperedge $\{2, 3, 4, 5, 6\}$ cannot be induced bisected. If both 1 and 6 are colored, we must color exactly two points out of $\{2, 3, 4, 5\}$. If $\{2, 3\}$ or $\{2, 4\}$ or $\{2, 5\}$ is colored, then $\{3, 4, 5\}$ is not induced bisected. Similarly, in each of the cases when two points out of $\{3, 4, 5\}$ is colored, some odd interval cannot be induced bisected. If both 1 and 6 remain uncolored, the intervals $\{2, 3, 4\}$ and $\{3, 4, 5\}$ are not induced bisected.

- $|S_1| = 5$: If all the colored points are consecutive, then the interval corresponding to the colored points cannot be induced bisected. Otherwise, there are three colored points which are consecutive, either followed or preceded by an uncolored point. The interval corresponding to these three consecutive colored points cannot be induced bisected.

This completes the proof of $\beta^d(\mathcal{I}_o) > 1$ when $n = 6$. In what follows, we generalize the arguments for arbitrary n.

Lemma 4.23 *$\beta^d(\mathcal{I}_o) > 1$, for $n \geq 5$ for arbitrary values of d.*

Proof. Assume for the sake of contradiction that $\beta^d(\mathcal{I}_o) = 1$ for some $n \geq 5$ and let Y denote the bicoloring that induced bisect all odd intervals of $\mathcal{I}_o$. Then, for any i, $2 \leq i \leq n - 1$, exactly two points out of the set $\{i - 1, i, i + 1\}$ must be colored in Y; otherwise, the interval $\{i - 1, i, i + 1\}$ is not induced bisected. Moreover, if some interval $\{i, i + 1, \ldots, j\}$ is colored by Y and $\{j + 1, j + 2\}$ (or $\{i - 2, i - 1\}$) remains uncolored, then $\{j, j + 1, j + 2\}$ or $\{i - 2, i - 1, i\}$ is not induced bisected. These two observation enforce that Y is either of type Y^{S_1} or Y^{S_2} given by Construction 1 or Construction 2, respectively. In either case, it is not hard to verify that some intervals are never induced bisected. □

From Lemma 4.22 and 4.23, we get the following theorem.

Theorem 4.24 *$\beta^d(\mathcal{I}_o) = 2$, for $n \geq 5$, when d is unrestricted.*

In what follows, we estimate $\beta^d(\mathcal{I}_e)$, where $\mathcal{I}_e$ denotes the family of all the even intervals of $[n]$.

Bounds for $\beta^d(\mathcal{I}_e)$

Theorem 4.25 $\frac{n-1}{d-1} \leq \beta^d(\mathcal{I}_e) \leq \frac{n}{d}(1 + \frac{2}{d} + o(1))$.

Proof. In order to establish a lower bound, observe that we have to cover all the intervals of size 2, and with d vertices, we can cover at most $d-1$ pairs of size 2. So, we obtain a lower bound of $\frac{n-1}{d-1}$ for $\beta^d(\mathcal{I}_e)$.

In order to establish an upper bound on $\beta^d(\mathcal{I}_e)$, we consider the case when d is even and d divides n. Color the vertices $S_1 = 1, \ldots, d$ with alternating +1 and -1's – let this coloring be Y^{S_1}. Note that (i) all the even intervals starting at any one of $\{1, 3, \ldots, d-1\}$ are induced bisected by Y^{S_1}; (ii) all the even intervals starting at any one of $\{2, 4, \ldots, d\}$ are either induced bisected by Y^{S_1} or contain both $\{d, d+1\}$. For example, let $[n] = \{1, 2, \ldots, 8\}$ and $d = 4$. Then, $Y_1 : \{1, 3\} \to +1, \{2, 4\} \to -1$. Note that the only even intervals starting at any one of $\{1, 2, 3, 4\}$ that are not induced bisected by Y_1 are $\{2, 3, 4, 5\}$, $\{2, 3, 4, 5, 6, 7\}$, $\{4, 5\}$,$\{4, 5, 6, 7\}$; they all contain $\{4, 5\}$.

So, after repeating such colorings for $\frac{n}{d}$ disjoint subsets ($S_1 = \{1, \ldots, d\}, S_2 = \{d+1, \ldots, 2d\}, \ldots, S_{\frac{n}{d}} = \{n-d+1, \ldots, n\}$), the only intervals that are not induced bisected by $Y^{S_1}, \ldots, Y^{S_{\frac{n}{d}}}$ contains one or more of $T = \{\{d, d+1\}, \{2d, 2d+1\}, \ldots, \{n-d, n-d+1\}\}$ as subsets. Choosing $\frac{d}{2}$ sets of T per bicoloring, taking their union and coloring i and $i+1$ with different colors, we need at most $\frac{2n}{d^2}$ extra bicolorings in this case. So, the total number of bicolorings is $\frac{n}{d}(1 + \frac{2}{d})$. In all other case ($d$ odd or d does not divide n), the number of extra bicolorings required is $o(\frac{n}{d})$. So, the theorem follows. $\square$

Chapter 5

The inverse problem: unbiased representative family for a set of bicolorings

5.1 Introduction

Let $\mathcal{B}$ denote a set of bicolorings of $[n] = \{1, \dots, n\}$, where each bicoloring $B \in \mathcal{B}$ maps each point $x \in [n]$ to either -1 or +1. Let Y_B denote the n-dimensional vector representing the bicoloring B, i.e. $Y_B = (B(1), \dots, B(n))$. A non-empty set $A \subseteq [n]$ is said to be an *unbiased representative* for a bicoloring $B \in \mathcal{B}$ if $\langle X_A, Y_B \rangle = 0$, where X_A denotes the 0–1 n-dimensional incidence vector corresponding to A. We call a family $\mathcal{A}$ of subsets of $[n]$ a *system of unbiased representatives* (or 'SUR') for $\mathcal{B}$ if for every bicoloring $B \in \mathcal{B}$, there exists at least one set $A \in \mathcal{A}$ such that $\langle X_A, Y_B \rangle = 0$. Note that the two monochromatic bicolorings can never have any unbiased representatives - we call these bicolorings 'trivial'. Let $\gamma(\mathcal{B})$ denote the minimum cardinality of a system of unbiased representatives for $\mathcal{B}$. We define the maximum of $\gamma(\mathcal{B})$ over all possible families $\mathcal{B}$ of non-trivial bicolorings of $[n]$ as $\gamma(n)$. Note that no singleton set of $[n]$ is a member of any optimal system of unbiased representatives. In the questions related to bisecting families and D-secting families, we are given a family $\mathcal{A}$ of subsets of $[n]$ and the problem is to compute a set of bicolorings $\mathcal{B}$ such that for every $A \in \mathcal{A}$, there exists

Individual

Attributes

m \ n	1	2	3	4	5	6	7
Age> 65?	-1	-1	1	1	1	1	1
Wt > 55?	1	-1	1	-1	1	1	1
Ht > $5ft$?	1	1	1	1	-1	-1	-1
•							

Figure 5.1: $A = \{1, 2, 3, 4\}$ is an unbiased representative for attributes age and weight, but not height.

a $B \in \mathcal{B}$ with $\langle X_A, Y_B \rangle = 0$. It is natural to ask the *inverse* problem: given a set of bicolorings $\mathcal{B}$, finding a small family of subsets satisfying the zero-dot-product property. This yields the notion of unbiased representation. This problem has application in the field of drug testing and formation of unbiased committees as discussed below.

Unbiased representatives are useful in testing products such as drugs over a large population where the effectiveness (or side effect) of a new drug is studied in correlation with a large set of patient attributes such as body weight, height, age, etc. Complementary extremes in the attributes, such as being obese or underweight, tall or short, and young or old, are relevant is such correlation studies. Such studies require patients with complementary ranges of values of a certain attribute to be present in equal (or roughly equal) numbers in the representative group for that attribute – such a group may be deemed to be an unbiased representative for the attribute. See Figures 5.1 and 5.2 for an illustration. However, selecting a separate sample of individuals for each attribute having equal representation of the complementary traits is practically impossible. So, one needs to select a family $\mathcal{A}$ of samples of individuals such that for any attribute B, there exists a sample $A \in \mathcal{A}$ which has an equal representation of individuals from the complementary traits of B. It is in the best interest to choose a family $\mathcal{A}$ of such groups of representatives of the smallest possible cardinality. It is not hard to see the direct

Individual

Attributes

m \ n	1	2	3	4	5	6	7
Age> 65?	-1	-1	1	1	1	1	1
Wt > 55?	1	-1	1	-1	1	1	1
Ht > $5ft$?	1	1	1	1	-1	-1	-1
•							

Figure 5.2: $B = \{2, 4, 5, 6\}$ is an unbiased representative for attributes weight and height, but not age.

mapping of this problem to the problem addressed in this paper. In a generic setting, SURs are useful in various applications where a collection of items (like individual patients) have many attributes (like weight, height and age), where the objective is to form a small collection of subsets of items with almost equal representation of opposite or complementary traits for each attribute.

5.1.1 Definitions and notations

We use 'SUR' to denote the phrase 'system of unbiased representatives'. For integers n and p, let $[n]$ denote the set $\{1, \dots, n\}$, and $[n \pm p]$ denote the set $\{n - p, n - p + 1, \dots, n + p\}$. A bicoloring B of $[n]$ is called a *k-bicoloring* if the number of +1's in B is exactly k. For a bicoloring $B : [n] \to \{-1, 1\}$, we use $B(+1)$ (respectively, $B(-1)$) to denote the set of points receiving color +1 (respectively, -1) under B. We use Y_B (X_A) to denote the n-dimensional ± 1 vector (respectively, 0–1 vector) representing the bicoloring B (respectively, $A \subseteq [n]$), i.e. $Y_B = (B(1), \dots, B(n))$. Note that $\langle Y_B, X_A \rangle = 0$ for some $A \in \binom{[n]}{r}$ implies that that r is even. Throughout the rest of the paper, we consider only the non-trivial bicolorings and assume that every set in a SUR is of even cardinality. Let $\gamma(\mathcal{B}, k, r)$ (respectively, $\gamma(\mathcal{B}, [k_1, k_2], [r_1, r_2])$) be the minimum cardinality of a SUR $\mathcal{A}$ for $\mathcal{B}$, where (i) each $B \in \mathcal{B}$ is a bicoloring of $[n]$ consisting of exactly k +1's (respectively, at least k_1 and at most k_2 +1's), and, (ii) each $A \in \mathcal{A}$ is an r-sized (respectively, at least r_1-sized and at most r_2-sized) subset of $[n]$. We define $\gamma(n, k, r)$ ($\gamma(n, [k_1, k_2], [r_1, r_2])$) as follows.

$$\gamma(n, k, r) = \max_{\mathcal{B}} \gamma(\mathcal{B}, k, r).$$

$$\gamma(n, [k_1, k_2], [r_1, r_2]) = \max_{\mathcal{B}} \gamma(\mathcal{B}, [k_1, k_2], [r_1, r_2]).$$

Since no singleton set of $[n]$ can be a member of any optimal system of unbiased representative and the monochromatic bicolorings, consisting of exactly zero (or n) +1's, are trivial, $\gamma(n, [1, n-1], [2, n])$ is the same as $\gamma(n)$.

5.1.2 Chapter outline

This chapter is divided into three logical sections. The first section (Section 5.2) focuses on obtaining $O(\log |\mathcal{B}|)$ upper bounds for SURs when (i) the collection $\mathcal{B}$ of bicolorings is unrestricted or has minor restrictions, and (ii) the sets in the SURs are unrestricted or have minor restrictions. When $\mathcal{B}$ consists of all the $2^n - 2$ non-monochromatic bicolorings, it is not difficult to show that $\frac{n}{2} \leq \gamma(\mathcal{B}, [1, n-1], [2, n]) \leq n - 1$. Using an application of Combinatorial Nullstellensatz [3], we improve the above lower bound to $n - 1$.

Theorem 5.1 *Let n be a positive integer and $k \in [n]$. Then, $\gamma(n, [1, n-k], [2, n]) = n - 1$, where $1 \leq k \leq \lceil \frac{n}{2} \rceil$.*

We relate the problem of SUR to the hitting set problem, which in turn implies relations with 'VC-dimension' provided $\epsilon n \leq |B(+1)| \leq (1-\epsilon)n$ for each $B \in \mathcal{B}$. For such families $\mathcal{B}$, this relationship assists in establishing an $O(\log |\mathcal{B}|)$ upper bound for cardinalities of any optimal SUR. Under a similar restriction for each $B \in \mathcal{B}$, if it is mandatory that each set in the SUR is of cardinality exactly r, the best upper bound obtained is large ($\Omega(\sqrt{r} \log |\mathcal{B}|)$). In order to establish an $O(\log |\mathcal{B}|)$ upper bound for size of an optimal SUR under this restriction, we introduce some error in the representations and we have the following theorem.

Theorem 5.2 *Let $r' \in [r \pm \lceil \frac{r}{2} \rceil]$, where $r \geq 8$ is an integer. Let $\mathcal{B}$ denote the set of all bicolorings $B \in \{-1, +1\}^n$, where $|B(+1) - B(-1)| \leq d$, for some $d \in \mathbb{N}$. Then, with high probability, one can construct a family $\mathcal{A}$ of cardinality at most $\ln |\mathcal{B}|$ in $O(n|\mathcal{B}| \ln |\mathcal{B}|)$ time consisting of r'-sized subsets such that for every $B \in \mathcal{B}$, there exists a set $A \in \mathcal{A}$ with $| \langle Y_B, X_A \rangle | \leq e\sqrt{r} + \frac{dr}{n}$.*

In the second part of the chapter (Section 5.3), we study the SUR problem where each $B \in \mathcal{B}$ is restricted to have exactly k +1's and each set in the SUR is required to be of cardinality exactly r, for some $r, k \in [n]$, $2 \leq r \leq 2k$. We relate the SUR problem under such restrictions to 'covering' problems, that enables us to use a deterministic algorithm of Lovász [44] and Stein [70] to compute such a SUR in polynomial time. In particular, for sufficiently large values of n, and $k \leq \log_4 \log_4(n^{0.5-\epsilon})$, we use a result of Alon et al. [5, Corollary 1.3] to establish the following asymptotically tight bound on $\gamma(n, k, 2k)$.

Theorem 5.3 *For sufficiently large values of n,*

$$\frac{\binom{n}{k}}{\binom{2k}{k}} \leq \gamma(n, k, 2k) \leq \frac{\binom{n}{k}}{\binom{2k}{k}}(1 + o(1)),$$

provided $k \leq \log_4 \log_4(n^{0.5-\epsilon})$, for any $0 < \epsilon < 0.5$.

The problem of estimation of $\gamma(n, k, r)$ becomes interesting when $k = \frac{n}{2}$ - the reduction to coverings gives a lower and upper bound of $\max\left(\left\lceil \frac{n}{2r} \right\rceil, c_1\sqrt{\frac{r(n-r)}{n}}\right)$ and $O(n\sqrt{\frac{r(n-r)}{n}})$, respectively. For $r = f(n)$, where $f(n)$ is an increasing function in n, this establishes only sub-linear lower bounds for $\gamma(n, \frac{n}{2}, r)$. We use a vector space orthogonality argument combined with a theorem of Keevash and Long [39] to obtain a linear lower bound on $\gamma(n, k, r)$ under certain restrictions on n, k and r.

Theorem 5.4 *Let $r = 2c$ for any odd integer $c \in \{1, \ldots, \frac{n}{2}\}$. Let k be an even integer, where $\epsilon n < k < (1-\epsilon)n$ for some $0 < \epsilon < 0.5$. Then, $\gamma(n, k, r) \geq \delta n$, where $\delta = \delta(\epsilon)$ is some real positive constant.*

Combined with an upper bound construction given in Lemma 5.18, this establishes an asymptotically tight bound for $\gamma(n, \frac{n}{2}, \frac{n}{2})$, when $\frac{n}{2} \equiv 2 \pmod 4$.

In the third part of the chapter (Section 5.4), we obtain the following inapproximability result for computing optimal SURs by using a result of Dinur and Steurer [25] on the inapproximability of the hitting set problem.

Theorem 5.5 *Let n and m be integers and let $r \leq (1 - \Omega(1))\frac{\ln m}{4}$. Then, no deterministic polynomial time algorithm can approximate the system of unbiased representative problem for a family of m bicolorings on $[n]$ to within a factor $(1-\Omega(1))\frac{\ln m}{4r}$ of the optimal when each set chosen in the representative family is required to have its cardinality at most r, unless P=NP.*

5.2 When cardinalities of sets in the 'SUR' are unrestricted or semi-restricted

5.2.1 Bounds on $\gamma(n, [k, n-1], [2, n])$

Recall that $\gamma(n) = \max_{\mathcal{B}} \gamma(\mathcal{B})$, where $\gamma(\mathcal{B})$ is the cardinality of an optimal system of unbiased representative for $\mathcal{B}$. Observe that $\gamma(\mathcal{B}_1) \leq \gamma(\mathcal{B}_2)$ when $\mathcal{B}_1 \subseteq \mathcal{B}_2$. So, to establish bounds on $\gamma(n)$, it suffices to consider the set of all the $2^n - 2$ non-monochromatic bicolorings as $\mathcal{B}$ and establish bounds on $\gamma(\mathcal{B})$. We have the following proposition.

Proposition 5.6 *Let n be an integer and $k \in [n]$.*

(i) $\gamma(n, [k, n-1], [2, n]) = \gamma(n, [1, n-k], [2, n])$.

(ii) $\gamma(n, [1, n-k], [2, n]) = \gamma(n, [1, \lfloor \frac{n}{2} \rfloor], [2, n])$, *for any* $1 \leq k \leq \lceil \frac{n}{2} \rceil$.

(iii) $\gamma(n, [1, n-k], [2, n]) \leq n-1$, *for* $1 \leq k \leq n$.

(iv) $\frac{n}{2} \leq \gamma(n, 1, [2, n]) \leq \gamma(n, [1, n-k], [2, n])$, *for* $1 \leq k \leq n-1$.

Proof. (i) For any k-bicoloring B, any unbiased representative A for B is also an unbiased representative for the bicoloring B', where $B'(+1) = B(-1)$ and $B'(-1) = B(+1)$.

(ii) The proof follows from the proof of Statement (i) in Proposition 5.6.

(iii) Let $\mathcal{B}$ denote the set of all the $2^n - 2$ non-monochromatic bicolorings. It is not hard to see that $\mathcal{A} = \{\{1,2\}, \{1,3\}, \ldots, \{1,n\}\}$ is a SUR of cardinality $n-1$ for $\mathcal{B}$.

(iv) Let $\mathcal{B} = \{B || B(+1)| = 1\}$. So, $|\mathcal{B}| = n$. For any $B \in \mathcal{B}$, if for any $A \subseteq [n]$, $\langle Y_B, X_A \rangle = 0$, then $|A| = 2$. Moreover, for any $A \in \binom{[n]}{2}$, exactly two $B \in \mathcal{B}$ has $\langle Y_B, X_A \rangle = 0$. So, we need at least $\frac{n}{2}$ two sized sets to form a SUR for $\mathcal{B}$. The second inequality follows from the containment. $\square$

In the construction leading to the proof of Statement (iii) in Proposition 5.6, only two-sized sets are used as unbiased representatives. We have the following slightly non-trivial construction assuming $n = 2^p$, for some integer p, giving similar bounds. Let $\mathcal{A}_2 = \{\{1,2\}, \{3,4\}, \ldots, \{n-1, n\}\}$: a partition of $[n]$ into two-sized sets. Let $\mathcal{A}_4 = \{\{1,2,3,4\}, \{5,6,7,8\}, \ldots, \{n-3, n-2, n-1, n\}\}$: a partition of $[n]$ into four-sized sets taken in that order. Similarly, repeating the construction for $p-2$ more steps, we obtain a sequence of partitions of $[n]$, $\mathcal{A}_2, \mathcal{A}_4, \ldots, \mathcal{A}_n$, where $\mathcal{A}_i$ is a partition of $[n]$ into i-sized $\frac{n}{i}$ parts, i.e., $\mathcal{A}_i = \{\{1, \ldots, i\}, \{i+1, \ldots, 2i\}, \ldots, \{n-i+1, \ldots, n\}\}$. Let $\mathcal{A} = \mathcal{A}_2 \cup \mathcal{A}_4 \cup \cdots \cup \mathcal{A}_n$. It follows that $|\mathcal{A}| = 2^{p-1} + 2^{p-2} + \ldots + 1 = 2^p - 1 = n-1$. To see that this is indeed a SUR for the set of all the $2^n - 2$ non-monochromatic bicolorings, let $B \in \{-1, 1\}^n$ denote any non-trivial bicoloring of $[n]$. Without loss of generality, assume that $|B(+1)| \leq |B(-1)|$. Let i ($2 \leq i \leq n$) be the minimum index such that there exists an $A \in \mathcal{A}_i$ with $A \setminus B(+1) \neq \phi$ and $A \cap B(+1) \neq \phi$. From construction of $\mathcal{A}_i$ and assumption on i, it follows that there exists consecutive parts $A_1, A_2 \in \mathcal{A}_{\frac{i}{2}}$ with $A_1 \subseteq B(+1)$, $A_2 \cap B(+1) = \phi$, and $A = A_1 \cup A_2$. So, it follows that A is an unbiased representative for B.

To establish a tight lower bound on $\gamma(n, [1, \lceil \frac{n}{2} \rceil], [2, n])$ ($\gamma(n, [1, n-1], [2, n])$), we need the following lemma.

Lemma 5.7 *Let $F \in \mathbb{F}(x_1, \ldots, x_n)$ be a polynomial and $S_1, \ldots, S_n$ be non-empty subsets of $\mathbb{F}$, for some field $\mathbb{F}$. If F vanishes on all but one point $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n \subseteq \mathbb{F}^n$, then deg(F) $\geq \sum_{i=1}^{n}(|S_i| - 1)$.*

Proof. For the sake of contradiction, assume that deg(F) $< \sum_{i=1}^{n}(|S_i| - 1)$. Consider the polynomials.

$$H(x_i) = \prod_{s \in S_i \setminus \{s_i\}} (x_i - s).$$
$$G(x_1, \ldots, x_n) = \prod_{i=1}^{n} H(x_i).$$

Note that deg(G) is $\sum_{i=1}^{n}(|S_i| - 1)$. Let $F(s_1, \ldots, s_n) = c_1$ and $G(s_1, \ldots, s_n) = c_2$. Then, the polynomial $c_2F - c_1G$ vanishes on all points of $S_1 \times \cdots \times S_n$. However, $c_2F - c_1G$ has degree $\sum_{i=1}^{n}(|S_i| - 1)$: the monomial $x_1^{|S_1|-1} \cdots x_n^{|S_n|-1}$ has $-c_1$ as its coefficient. Using Combinatorial Nullstellensatz [3], there exists at least one point in $S_1 \times \cdots \times S_n$ where $c_2F - c_1G$ is non-zero which is a contradiction. $\square$

Proof of Theorem 5.1

Statement of Theorem 5.1. Let n be a positive integer and $k \in [n]$. Then, $\gamma(n, [1, n-k], [2, n]) = n - 1$, where $1 \leq k \leq \lceil \frac{n}{2} \rceil$.

Proof. From Statements (ii) and (iii) of Proposition 5.6, we know that in order to prove Theorem 5.1, we only need to establish a lower bound of $n-1$ for $\gamma(n, [1, n-1], [2, n])$.

Let $\mathcal{B}$ denote the set of all the $2^n - 2$ non-monochromatic bicolorings of $[n]$. Let $\mathcal{A}$ be a SUR of minimum cardinality for $\mathcal{B}$. Let Y_B (X_A) denote the n-dimensional ± 1 vector (respectively, 0–1 vector) representing the bicoloring B (respectively, $A \subseteq [n]$) Consider the polynomial $P(Y_B)$, $B \in \mathcal{B}$.

$$P(Y_B) = \prod_{A \in \mathcal{A}} \langle X_A, Y_B \rangle. \tag{5.1}$$

From the definition of $\mathcal{A}$, $P(Y_B)$ vanishes on all non-trivial bicolorings of $[n]$. Now, consider the following polynomial $P'(X)$.

$$P'(X = (x_1, \ldots, x_n)) = P(Y_B = (1 - 2x_1, \ldots, 1 - 2x_n))(x_1 + \ldots + x_n - n). \tag{5.2}$$

$P'(X)$ vanishes at every $X \in \{0,1\}^n$ except at the point $(0, \ldots, 0)$: P vanishes at every $X \in \{0,1\}^n$ except the two points $(0, \ldots, 0)$ and $(1, \ldots, 1)$ and $(x_1 + \ldots + x_n - n)$ vanishes at $(1, \ldots, 1)$. $P'(X)$ has degree at most $deg(P) + 1$ (note that one can repeatedly replace x_i^2 with x_i since $x_i \in \{0,1\}$). Using Lemma 5.7 with each $S_i = \{0,1\}$, $1 \leq i \leq n$, it follows that $deg(P) + 1 \geq deg(P') \geq n$. So, $|\mathcal{A}| = deg(P) \geq n - 1$. □

Note that in Section 5.2.1, the underlying set $\mathcal{B}$ of all the non-trivial bicolorings of $[n]$, has cardinality $|\mathcal{B}| = 2^n - 2$. In this case, Theorem 5.1 establishes that $\gamma(n, [1, n-1], [2, n]) = n - 1 = \Theta(\log |\mathcal{B}|)$. In the following section, we match the $O(\log |\mathcal{B}|)$ upper bound for slightly restricted sets $\mathcal{B}$ of bicolorings.

5.2.2 Relation to hitting sets for arbitrary collection of bicolorings

Let $\mathcal{S}$ denote a collection of subsets of $[n]$. A subset $V \subseteq [n]$ is a *hitting set* for $\mathcal{S}$ if for every $S \in \mathcal{S}$, $V \cap S$ is non-empty. Let $H(\mathcal{S})$ denote a minimum cardinality hitting set of $\mathcal{S}$. The decision version of the Hitting set problem is: "Given the pair $(\mathcal{S}, [n])$ and an integer k as input, decide whether there exists a hitting set of cardinality at most k for $\mathcal{S}$".

Lemma 5.8 *Let* $\mathcal{B} = \{B_0, \ldots, B_{m-1}\} \subseteq \{-1, +1\}^n$ *be a family of bicolorings of* $[n]$.

Construct the family $\mathcal{C} = \{C_1, \ldots, C_{2m}\}$ *where* $C_{2i+1} = B_i(+1)$ *and* $C_{2i+2} = B_i(-1)$*, for* $0 \leq i \leq m-1$*. Let* $H = \{h_1, h_2, h_3, \ldots\}$ *denote a hitting set for* $\mathcal{C}$*. Define* $\mathcal{A} = \{(h_1, h_q) | h_q \in H, q > 1\}$*. Then,* $\mathcal{A}$ *is a SUR for* $\mathcal{B}$ *of cardinality* $|H| - 1$*.*

Proof. For the sake of contradiction, assume that $B_i \in \mathcal{B}$ has no unbiased representative in $\mathcal{A}$. Assume that $h_1 \in B_i(+1)$. Since H is a hitting set for $\mathcal{C}$, there exists some $h_q \in H$ such that h_q hits C_{2i+2} (and, thereby $B_i(-1)$). Then, the pair (h_1, h_q) is an unbiased representative for B_i, a contradiction to our assumption. So, $h_1 \notin B_i(+1)$. But this implies that $h_1 \in B_i(-1)$. A similar contradiction can be obtained in this case. □

Let $\mathcal{B}$ be restricted to a special family of bicolorings: the number of +1's for each $B \in \mathcal{B}$ lies in the range ϵn and $(1-\epsilon)n$, i.e., $\epsilon n \leq |B(+1)| \leq (1-\epsilon)n$, for some fixed $0 < \epsilon < \frac{1}{2}$. Construct the family $\mathcal{C}$ as above and let d be the VC-dimension of $\mathcal{C}$. Note that every $C \in \mathcal{C}$ has size at least ϵn, for some fixed $\epsilon < \frac{1}{2}$. Using a result of Haussler and Welzl [33] which was improved by Komlos et al. [40], we can get an 'epsilon net' H (which is a hitting set for $\mathcal{C}$) of cardinality at most $\frac{d}{\epsilon}(\ln\frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon} + 6)$ (see Corollary 15.6 of [55] for this exact bound). Using Lemma 5.8, it follows that we can construct a SUR for $\mathcal{B}$ of cardinality $\frac{d}{\epsilon}(\ln\frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon} + 6) - 1$. Since any family $\mathcal{C}$ of VC-dimension d has cardinality at least 2^d, this establishes an $O(\log|\mathcal{C}|) = O(\log|\mathcal{B}|)$ upper bound for the cardinality of any optimal SUR under no restriction on set sizes. We state the result as a proposition below.

Proposition 5.9 *Let* $0 \leq \epsilon \leq \frac{1}{2}$ *be a constant. Let* $\mathcal{B}$ *be a family of bicolorings, where* $\epsilon n \leq |B(+1)| \leq (1-\epsilon)n$*, for each* $B \in \mathcal{B}$*. Let* $\mathcal{C}$ *be the family constructed from* $\mathcal{B}$ *as in Lemma 5.8. Let* d *be the VC-dimension of* $\mathcal{C}$*. Then, we can construct a SUR for* $\mathcal{B}$ *of cardinality* $\frac{d}{\epsilon}(\ln\frac{1}{\epsilon} + 2\ln\ln\frac{1}{\epsilon} + 6) - 1$*.*

In both Section 5.2.1 and 5.2.2, the $O(\log|\mathcal{B}|)$ cardinality SURs contained sets of small sizes (2-sized sets) as well. In what follows, we study the problem of SURs made of large cardinality sets. In order to obtain a similar $O(\log|\mathcal{B}|)$ bound for such a SUR, we inevitably introduce some error in the representation.

5.2.3 Analysis with bias in representation

Consider the problem of estimation of $\gamma(\mathcal{B})$ for a set of bicolorings in terms of $|\mathcal{B}|$, where (i) the number of +1's in each $B \in \mathcal{B}$ lies in the range $\{\alpha n, \alpha n+1, \ldots, (1-\alpha)n\}$ for some $0 < \alpha < \frac{1}{2}$, and (ii) each set in the SUR is of cardinality exactly r, for some $2 \leq r \leq n$. Choosing r elements, namely $x_1, \ldots, x_r$, from $[n]$ independently and uniformly at random, the probability p that a fixed bicoloring $B \in \mathcal{B}$ does not have $\langle Y_B, X_A \rangle = 0$, where $A = \{x_1, \ldots, x_r\}$, is at most

$$1-\binom{r}{\frac{r}{2}}\left(\frac{\alpha n}{n}\right)^{\frac{r}{2}}\left(\frac{(1-\alpha)n}{n}\right)^{\frac{r}{2}} \leq 1-C\frac{2^r}{\sqrt{r}}\alpha^{\frac{r}{2}}(1-\alpha)^{\frac{r}{2}} < e^{-C\frac{2^r}{\sqrt{r}}\alpha^{\frac{r}{2}}(1-\alpha)^{\frac{r}{2}}}, \text{where } C = \frac{1}{\sqrt{\pi}}.$$

Let $\mathcal{A}$ be constructed by choosing t r-element sets into $\mathcal{A}$ independently, where each r-element set is chosen as described above. Using union bound, the probability that some $B \in \mathcal{B}$ has $\langle Y_B, X_A \rangle \neq 0$ for all $A \in \mathcal{A}$, is $|\mathcal{B}|(e^{-C\frac{2^r}{\sqrt{r}}\alpha^{\frac{r}{2}}(1-\alpha)^{\frac{r}{2}}})^t$. This gives an upper bound of $\frac{\sqrt{r}}{C2^r\alpha^{\frac{r}{2}}(1-\alpha)^{\frac{r}{2}}}\ln(|\mathcal{B}|)$ for $|\mathcal{A}|$. Using Proposition 5.11, the case when $k = \frac{n}{2}$ and $r = 2$ yields an asymptotically tight example for this upper bound. We have the following proposition.

Proposition 5.10 *Let $\mathcal{B}$ denote a set of bicolorings, where the number of +1's in each $B \in \mathcal{B}$ lie in the range $\{\alpha n, \alpha n+1, \ldots, (1-\alpha)n\}$ for some $0 < \alpha < \frac{1}{2}$. Let $\mathcal{A}$ denote a minimum cardinality SUR for $\mathcal{B}$, where each $A \in \mathcal{A}$ has cardinality exactly r. Then,*

$$|\mathcal{A}| \leq \frac{\sqrt{r}}{C2^r\alpha^{\frac{r}{2}}(1-\alpha)^{\frac{r}{2}}}\ln(|\mathcal{B}|), \tag{5.3}$$

where $C = \frac{1}{\sqrt{\pi}}$.

When $\alpha = \frac{1}{2} - \epsilon$, for some $0 \leq \epsilon < \frac{1}{2}$, Inequality 5.3 becomes

$$|\mathcal{A}| \leq \frac{\sqrt{r}}{C(1-4\epsilon^2)^{\frac{r}{2}}}\ln(|\mathcal{B}|). \tag{5.4}$$

Using the fact that $(1-\frac{1}{m+1})^m \geq \frac{1}{e}$, the right hand term is at most $e^{(\frac{4\epsilon^2}{1-4\epsilon^2})^{\frac{r}{2}}}\sqrt{\pi r}\ln|\mathcal{B}|$.

Therefore, when $r \in O(1)$, we have an $O(\ln|\mathcal{B}|)$ upper bound for any optimal SUR consisting of r sized sets for $\mathcal{B}$. However, if r is any increasing function in n, the upper bound given by Proposition 5.10 is large (even if $\epsilon = \frac{1}{n}$, the term $\frac{\sqrt{r}}{C(1-4\epsilon^2)^{\frac{r}{2}}}\ln(|\mathcal{B}|)$ is $\Omega(\sqrt{r}\ln|\mathcal{B}|)$). For large values of r, in order to obtain an $O(\ln(|\mathcal{B}|))$ upper bound for $|\mathcal{A}|$, one may allow some error in representation studied in the following section. Let $\mathcal{B}$ denote the set of all bicolorings $B \in \{-1,+1\}^n$, where $|B(+1) - B(-1)| \leq d$, for some $d \in \mathbb{N}$. Our problem is to find a small sized family $\mathcal{A}$ for $\mathcal{B}$ such that

1. each $A \in \mathcal{A}$ is reasonably large;
2. for every $B \in \mathcal{B}$, there exists a set $A \in \mathcal{A}$ such that $|\langle Y_B, X_A\rangle| \leq \Delta$, where $\Delta = \Delta(r,d,n)$ is as small as possible.

Proof of Theorem 5.2

Statement of Theorem 5.2. Let $r' \in [r \pm \lceil \frac{r}{2} \rceil]$, where $r \geq 8$ is an integer. Let $\mathcal{B}$ denote the set of all bicolorings $B \in \{-1,+1\}^n$, where $|B(+1) - B(-1)| \leq d$, for some $d \in \mathbb{N}$. Then, with high probability, one can construct a family $\mathcal{A}$ of cardinality at most $\ln|\mathcal{B}|$ in $O(n|\mathcal{B}|\ln|\mathcal{B}|)$ time consisting of r'-sized subsets such that for every $B \in \mathcal{B}$, there exists a set $A \in \mathcal{A}$ with $|\langle Y_B, X_A\rangle| \leq e\sqrt{r} + \frac{dr}{n}$.

Proof. We construct a set $A \subset [n]$ of size $r' \in [r \pm \lceil \frac{r}{2} \rceil]$ by picking each element of $[n]$ into A independently with probability $\frac{r}{n}$. Let $X_A = (a_1, \ldots, a_n)$ denote the corresponding random vector where each $a_i \in \{0,1\}$. Note that $|A| = \sum_{i=1}^{n} a_i$. So, using linearity of expectation, $(\mu =)\mathbb{E}[|A|] = \sum_{i=1}^{n}\mathbb{E}[a_i] = r$. Moreover, since a_i's are independent, $Var[|A|] = \sum_{i=1}^{n} Var[a_i] = r(1-\frac{r}{n})$. So, using the following form of Chernoff's bound $P(|X-\mu| > \Delta\mu) < (\frac{e^{\Delta}}{(1+\Delta)^{(1+\Delta)}})^{\mu} + (\frac{e^{-\Delta}}{(1-\Delta)^{(1-\Delta)}})^{\mu}$, we get, $P(|\sum_{i=1}^{n} a_i - r| > 0.5r) < 0.72$, for $r \geq 8$. So, we can sample a family $\mathcal{A}$ of cardinality t (t to be chosen later) consisting of sets of size $r' \in [r \pm \frac{r}{2}]$.

Let $B \in \mathcal{B}$ be a bicoloring, where $B(+1) - B(-1) = d_1$, where $-d \leq d_1 \leq d$. Let $Y_B = (b_1, \ldots, b_n)$ denote the corresponding bit vector, where each $b_i \in \{-1,+1\}$. Let $Y = \langle Y_B, X_A\rangle$. Since $Y = \sum_{i=1}^{n} a_i b_i$, Y becomes a random variable (note that $a_i b_i$ can take values $\{-1, 0, 1\}$ and are independent). So, $\mathbb{E}[Y] = \sum_{i=1}^{n} b_i \mathbb{E}[a_i] = \frac{d_1 r}{n}$. It follows that $Var[Y] = \sum_{i=1}^{n} b_i^2 Var[a_i] = r(1 - \frac{r}{n})$. So, using Chebyshev's inequality, we get, $P\left(|Y - \frac{d_1 r}{n}| \geq e\sqrt{r}\right) \leq \frac{1}{e^2}(1 - \frac{r}{n}) < \frac{1}{e^2}$. That is, the probability that $|\langle Y_B, X_A \rangle| > \frac{d_1 r}{n} + e\sqrt{r}$ is at most $\frac{1}{e^2}$. Let E denote the bad event that some $B \in \mathcal{B}$ has $|\langle Y_B, X_A \rangle| > \frac{dr}{n} + e\sqrt{r}$ for all $A \in \mathcal{A}$. Using union bound, $P(E) \leq |\mathcal{B}|(\frac{1}{e^2})^t$. Setting $|\mathcal{B}|(\frac{1}{e^2})^t$ to at most $\frac{1}{2}$, we get, $t \geq \ln |\mathcal{B}|$.

Independently choose $100t$ subsets of $[n]$ (call this collection $\mathcal{D}$), where each $D \in \mathcal{D}$ is constructed by picking an element of $[n]$ independently with probability $\frac{r}{n}$. Let $\mathcal{C} \subseteq \mathcal{D}$ be the sub-collection of r'-sized subsets in $\mathcal{D}$, where $r' \in \lceil \frac{r}{2} \rceil$. Then, $E[|\mathcal{C}|] \geq 28t$. Since $Var[|\mathcal{C}|] \leq 25t$, with high probability, $|\mathcal{C}| \geq 10t$. Partition $\mathcal{C}$ into t-sized sets. With high probability, one of the parts will form our desired family $\mathcal{A}$ that is a SUR (with restricted error) for $\mathcal{B}$. $\square$

Comparison between Theorem 5.2 and Proposition 5.10: Expressing d in Theorem 5.2 in terms of α in Proposition 5.10, $(1 - 2\alpha)n = d$. So, $\epsilon = \frac{1}{2} - \alpha = \frac{d}{2n}$. Substituting this value of ϵ in Inequality 5.4, we get a SUR of cardinality $\Omega(\sqrt{r} \ln |\mathcal{B}|)$ with no error for $\mathcal{B}$.

5.3 When cardinalities of sets in the 'SUR' and +1's in the bicolorings are restricted

For any k-bicoloring B of $[n]$, and any $A \subseteq [n]$, if A is an unbiased representative for B, then $2 \leq |A| \leq 2k$: otherwise, $\langle Y_B, X_A \rangle \neq 0$. Recall that $\gamma(n, k, r) = \gamma(\mathcal{B})$, where (i) $\mathcal{B}$ is the collection of the $\binom{[n]}{k}$ distinct k-bicolorings, (ii) $\gamma(\mathcal{B})$ is the cardinality of an optimal SUR $\mathcal{A}$ for $\mathcal{B}$, and, (iii) each $A \in \mathcal{A}$ has cardinality exactly r. We have the following propositions.

Proposition 5.11 $\max(\lceil \frac{n-k}{r} \rceil, \lceil \frac{k}{r} \rceil) \leq \gamma(n, k, r)$.

Proof. Consider the case when $k \leq \lfloor \frac{n}{2} \rfloor$. It is not hard to see that with $\lfloor \frac{n-k}{r} \rfloor$ disjoint r-sized subsets, there exists a k-sized subset (say, S) of $[n]$ that is completely disjoint from the union of these r-sized subsets. The bicoloring with the points in S colored +1 and the points in $[n] \setminus S$ colored -1 does not have any unbiased representative among the disjoint r-sized subsets. $\square$

Proposition 5.12 $\frac{2}{r(r-1)}\gamma(n, k-1, r-2) \leq \gamma(n, k, r) \leq (n-r+1)\gamma(n, k-1, r-2)$, *for* $r \geq 4$.

Proof. Let $\mathcal{B}_i$ denote the set of all the bicolorings consisting of exactly i +1's, for $i \in \{k, k-1\}$. Let $\mathcal{A}_{r-2}$ denote a family of $(r-2)$-sized subsets that is an optimal unbiased representative family for $\mathcal{B}_{k-1}$. For any $A \in \mathcal{A}_{r-2}$, let $\bar{A} = [n] \setminus A = \{x_1, \ldots, x_{n-r+2}\}$. For each $A \in \mathcal{A}_{r-2}$, we construct $(n-r+1)$ r-sized subsets as follows: $A^1 = A \cup \{x_1, x_2\}$, $A^2 = A \cup \{x_1, x_3\}$, $\cdots$, $A^{n-r+1} = A \cup \{x_1, x_{n-r+2}\}$. Let $\mathcal{A}_r = \cup_{A \in \mathcal{A}_{r-2}}\{A^1, \cdots, A^{n-r+1}\}$. To see that $\mathcal{A}_r$ is a system of unbiased representative for $\mathcal{B}_k$, consider any $B \in \mathcal{B}_k$ and a $(k-1)$-sized subset $B' \subset B_k$. Let $A' \in \mathcal{A}_{r-2}$ has $\langle Y_{B'}, X_{A'} \rangle = 0$. From the construction, it follows that there is at least one $A \in \{A'^1, \cdots, A'^{n-r+1}\}$ such that $\langle Y_B, X_A \rangle = 0$.

For the lower bound, consider a SUR $\mathcal{A}$ for $\mathcal{B}_k$ of size $\gamma(n, k, r)$. For each $A \in \mathcal{A}$, let $\mathcal{F}_A$ denote the family of $\binom{r}{r-2}$ distinct $(r-2)$-sized subsets of A. Then, $\mathcal{A}' = \cup_{A \in \mathcal{A}} F_A$ is an unbiased representative family for $\mathcal{B}_{k-1}$ where each set in the family is of size exactly $(r-2)$. $\square$

A simple averaging argument gives the following lower bound.

$$\gamma(n, k, r) \geq \frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}}. \tag{5.5}$$

To establish an upper bound, we reduce this problem to a covering problem and then make use of a result by Lovász and Stein [70, 44].

Definition 5.13 *Given a family $\mathcal{F}$ of subsets of some finite set X, the cover number $Cov(\mathcal{F})$ of $\mathcal{F}$ is the minimum number of members of $\mathcal{F}$ whose union includes all the points in X.*

Theorem 5.14 *[70, 44, 36] If each member of $\mathcal{F}$ covers at most a elements and each element in X is covered by at least v members of $\mathcal{F}$, then $Cov(\mathcal{F}) \leq \frac{|\mathcal{F}|}{v}(1+\ln a)$.*

We have the following theorem.

Theorem 5.15 *Let n be an integer, $r, k \in [n]$, $2 \leq r \leq 2k$ and r is even. Then,*

$$\frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}} \leq \gamma(n,k,r) \leq \frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}}\left(1+0.7r+\ln(\binom{n-r}{k-\frac{r}{2}})\right).$$

Proof. Consider the following construction of a uniform family of subsets based on the $\binom{n}{[k]}$ distinct k-bicolorings and $\binom{n}{r}$ distinct r-sized subsets of $[n]$.

Construction 3 *Corresponding to each distinct k-bicoloring B in $\binom{[n]}{k}$, we add a point v_B to X. Corresponding to each distinct r-sized subset A in $\binom{[n]}{r}$, we add a set e_A to $\mathcal{F}$, where e_A is the collections of all v_B's such that $\langle X_A, Y_B\rangle = 0$. So, e_A 'covers' v_B if and only if $v_B \in e_A$.*

So, $|X| = \binom{n}{k}$, $|\mathcal{F}| = \binom{n}{r}$. Clearly, $a = \binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}$, $v = \binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}}$. It follows from the construction that $\gamma(n,k,r) \leq Cov(\mathcal{F})$. So, from Theorem 5.14, we have

$$\gamma(n,k,r) \leq \frac{\binom{n}{r}}{\binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}}}\left(1+\ln(\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}})\right). \tag{5.6}$$

Double counting (B, A) pairs, where B is a k-bicoloring and A is a r-sized subset that covers B, we get

$$\binom{n}{k}\binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}} = \binom{n}{r}\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}. \tag{5.7}$$

Combining Inequalities 5.6 and 5.7, and from Inequality 5.5, Theorem 5.15 follows.

□

Since Lovász-Stein method is deterministic and constructive, the above reduction gives a deterministic polynomial time algorithm for obtaining a SUR. Moreover, from Theorem 5.15, it follows that $\gamma(n,k,r)$ is $O(k \ln n)$ approximable ($k + 0.2r + (k - \frac{r}{2}) \ln(\frac{n-r}{k-\frac{r}{2}})$ to be precise) and when $k = \frac{r}{2}$, the approximation factor becomes $O(r)$ ($1 + 0.7r$ to be exact). However, if $k \leq \log_4 \log_4(n^{0.5-\epsilon})$ and $r = 2k$, for some $0 < \epsilon < 0.5$, then this upper bound can be improved further.

5.3.1 Tight upper bounds under more restrictions

From Construction 3, it is clear that the approximation factor for $\gamma(n,k,r)$ in Theorem 5.15 comes as a consequence of the approximation factor for the cover number given by Lovász-Stein Theorem. So, tighter bounds for the cover number should translate into tighter bounds for $\gamma(n,k,r)$. Let $v(B,D)$ denote the number of r-sized sets that are unbiased representatives for both B and D, for any pair (B,D) of k-bicolorings, where $B \neq D$. Let $v_{pair} = \max\limits_{\substack{B,D \in \binom{[n]}{k},\\ B \neq D}} v(B,D)$. Rödl nibble method [61, 6] establishes asymptotically tight bounds for the cover number provided the uniformity a of the family $\mathcal{F}$ in Construction 3 is fixed, $v \to \infty$, and $v_{pair} \in o(v)$. Alon et al. [5] relaxed the condition to $a = o(\log v)$ provided $v_{pair} \in o(\frac{v}{e^{2a} \log v})$. In the estimation of $\gamma(n,k,r)$, if $k \leq \log_4 \log_4(n^{0.5-\epsilon})$ and $r = 2k$, for any $0 < \epsilon < 0.5$, using Construction 3, it follows that $a < 2^r \in O(\log n)$ and $\log n \in o(\log v)$. So, in order to prove Theorem 5.3, it suffices to show that $v_{pair} \in o(\frac{v}{e^{2a} \log v})$.

Lemma 5.16 *$v_{pair} \in o(\frac{v}{e^{2a} \log v})$, when $r = 2k$ and $k \leq \log_4 \log_4(n^{0.5-\epsilon})$, for any $0 < \epsilon < 0.5$.*

Proof. In order to prove the lemma, it is important to note that $v(B,D)$ depends intrinsically on the cardinality of $B(+1) \cap D(+1)$. Let S be some r-sized subset of $[n]$. Let $i_B = S \cap (B(+1) \setminus D(+1))$, $i_D = S \cap (D(+1) \setminus B(+1))$, $j_{BD} = S \cap (B(+1) \cap D(+1))$ and $j_{\overline{BD}} = S \cap ([n] \setminus (B(+1) \cup D(+1))$ (see Figure 5.3). So, $S = i_B \cup i_D \cup j_{BD} \cup j_{\overline{BD}}$. If S is an unbiased representative for B, then $|i_B| + |j_{BD}| = |i_D| + |j_{\overline{BD}}| = \frac{r}{2}$. If S

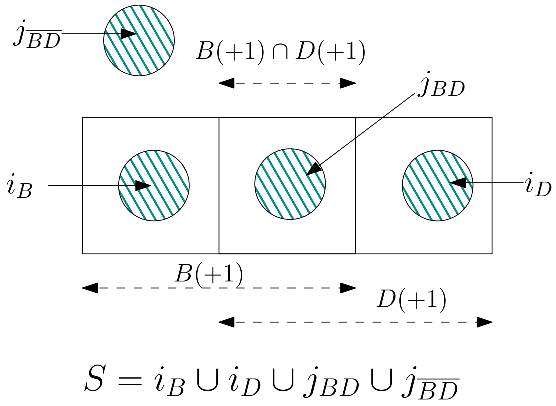

Figure 5.3: S is some r-sized subset of $[n]$. Let $i_B = S \cap (B(+1) \setminus D(+1))$, $i_D = S \cap (D(+1) \setminus B(+1))$, $j_{BD} = S \cap (B(+1) \cap D(+1))$ and $j_{\overline{BD}} = S \cap ([n] \setminus (B(+1) \cup D(+1))$. So, $S = i_B \cup i_D \cup j_{BD} \cup j_{\overline{BD}}$. If S is an unbiased representative for B, then $|i_B| + |j_{BD}| = |i_D| + |j_{\overline{BD}}|$. If S is an unbiased representative of D, then $|i_D| + |j_{BD}| = |i_B| + |j_{\overline{BD}}|$. So, if S is an unbiased representative of both B and D, then $|i_B| = |i_D|$ and $|j_{BD}| = |j_{\overline{BD}}|$.

is an unbiased representative of D, then $|i_D| + |j_{BD}| = |i_B| + |j_{\overline{BD}}| = \frac{r}{2}$. Therefore, if S is an unbiased representative of both B and D, then (i) $|i_B| = |i_D|$ ($= i$, say), (ii) $|j_{BD}| = |j_{\overline{BD}}|$ ($= j$, say), and (iii) $2i + 2j = r = 2k$. Let $x = |B(+1) \cap D(+1)|$. We have,

$$v(B, D) = \sum_{\substack{i,j: j \leq x, \\ i \leq k-x, \\ i+j = \frac{r}{2}}} \binom{x}{j} \binom{n - 2k + x}{j} \left(\binom{k-x}{i} \right)^2. \tag{5.8}$$

Since $|B(+1)| = |D(+1)| = k$, applying Condition (iii), we get $x = j$ and $k - x = i$. In other words, if S is an unbiased representative of cardinality $r = 2k$ for both the k-bicolorings B and D, $B(+1) \cup D(+1) \subseteq S$. So, for any pair B, D of k-bicolorings, exactly one term in the summation of Equation 5.8 remains valid, namely $\binom{x}{x}\binom{n-2k+x}{x}\left(\binom{k-x}{k-x}\right)^2$. For instance, when $x = k-1$, $v(B, D) = \binom{n-k-1}{k-1}$; when $x = k-2$, $v(B, D) = \binom{n-k-2}{k-2}$, etc. Therefore, $\frac{v(B,D)}{v(B',D')} = \Omega(\frac{n}{k})$ if $|B(+1) \cap D(+1)| = k-1$ and $|B'(+1) \cap D'(+1)| \leq k-2$. So, $v_{pair} = v(B, D)$, when $|B(+1) \cap D(+1)| = k-1$

provided $r = 2k$. Thus, $v_{pair} = \binom{n-k-1}{\frac{r}{2}-1}$, when $r = 2k$. Computing $\frac{v_{pair}}{v}$,

$$\frac{v_{pair}}{v} = \frac{\binom{n-k-1}{\frac{r}{2}-1}}{\binom{k}{\frac{r}{2}}\binom{n-k}{\frac{r}{2}}} = \frac{r}{2(n-k)}. \tag{5.9}$$

Note that $\log v = O(r \log n)$, $e^{2a} \leq n^{1-2\epsilon}$ since $k \leq \log_4 \log_4(n^{0.5-\epsilon})$. So, $\frac{v_{pair}e^{2a}\log v}{v} = O(\frac{r^2 \log n}{n^{2\epsilon}}) \to 0$, when $n \to \infty$.

Proof of Theorem 5.3

Statement of Theorem 5.3. For sufficiently large values of n,

$$\frac{\binom{n}{k}}{\binom{2k}{k}} \leq \gamma(n, k, 2k) \leq \frac{\binom{n}{k}}{\binom{2k}{k}}(1 + o(1)),$$

provided $k \leq \log_4 \log_4(n^{0.5-\epsilon})$, for any $0 < \epsilon < 0.5$.

Proof. From Lemma 5.16, and using the result of Alon et al. [5, Corollary 1.3] to obtain coverings, the proof follows. □

5.3.2 $\gamma(n, k, r)$, when $k = n/2$

Let $\mathcal{B}$ denote the set of all $\binom{n}{\frac{n}{2}}$ distinct $\frac{n}{2}$-bicolorings. It is not hard to see that $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \ldots, \{1, \frac{n}{2} + 1\}\}$ is a SUR of cardinality $\frac{n}{2}$ for $\mathcal{B}$. Together with Proposition 5.11, this establishes $\frac{n}{4} \leq \gamma(n, \frac{n}{2}, 2) \leq \frac{n}{2}$. It is easy to see that $\gamma(n, \frac{n}{2}, n) = 1$. For arbitrary values of r, from Theorem 5.15 and Proposition 5.11, we have,

$$\max\left(\left\lceil \frac{n}{2r} \right\rceil, c_1\sqrt{\frac{r(n-r)}{n}}\right) \leq \gamma(n, \frac{n}{2}, r) \leq c_2 n\sqrt{\frac{r(n-r)}{n}}, \text{ where } c_1 \text{ and } c_2 \text{ are constants.} \tag{5.10}$$

When $r = \frac{n}{2}$, this establishes a lower bound and upper bound of $\Omega(\sqrt{n})$ and $O(n\sqrt{n})$, respectively. In general, when $r = f(n)$ is an increasing function in n, this establishes sub-linear lower bounds for $\gamma(n, \frac{n}{2}, r)$.

We use an extension of a theorem of Frankl and Rödl [29] given by Keevash and Long [39] to obtain a linear lower bound on $\gamma(n, k, r)$ under certain restrictions on k and r. Let $\mathcal{D} \subseteq [q]^n$ be a q-ary code. For any $x, y \in \mathcal{D}$, the Hamming distance between x and y is the number of indices where $x(i) \neq y(i)$, for $1 \leq i \leq n$. The code $\mathcal{D}$ is called *d-avoiding* if the Hamming distance between no pair of code-words in $\mathcal{D}$ is d. The following upper bound for d-avoiding codes is given in [39].

Theorem 5.17 *[39] Let $\mathcal{D} \subseteq [q]^n$ and let ϵ satisfy $0 < \epsilon < \frac{1}{2}$. Suppose that $\epsilon n < d < (1-\epsilon)n$ and d is even if $q = 2$. If $\mathcal{D}$ is d-avoiding, then $|\mathcal{D}| \leq q^{(1-\delta)n}$, for some positive constant $\delta = \delta(\epsilon)$.*

We have the following lower bound for $\gamma(n, k, r)$, when $r = 2c$ for any odd integer $c \in \{1, \dots, \frac{n}{2}\}$ and $\epsilon n < k < (1-\epsilon)n$, for some $0 < \epsilon < 0.5$.

Proof of Theorem 5.4

Statement of Theorem 5.4. Let $r = 2c$ for any odd integer $c \in \{1, \dots, \frac{n}{2}\}$. Let k be an even integer, where $\epsilon n < k < (1-\epsilon)n$ for some $0 < \epsilon < 0.5$. Then, $\gamma(n, k, r) \geq \delta n$, where $\delta = \delta(\epsilon)$ is some real positive constant.

Proof. Let $\mathcal{B} = \{B_1, \dots, B_{\binom{n}{k}}\}$ denote the set of all the bicolorings of $[n]$ consisting of exactly k +1's. We construct a family $\mathcal{C} = \{C_1, \dots, C_{\binom{n}{k}}\}$, where C_i corresponds to the +1 colored points of $B_i \in \mathcal{B}$. Let $\mathcal{A}$ be a SUR for $\mathcal{B}$, where each $A \in \mathcal{A}$ has cardinality exactly $2c$ for some odd number $c \in [n]$. Note that $\langle Y_{B_i}, X_A \rangle = 0$ implies that $\langle X_{C_i}, X_A \rangle = c$, where X_{C_i} denotes the 0–1 incidence vector corresponding to the set C_i. Let $V \subset \{0,1\}^n$ denote the vector space spanned by the vectors X_A's, $A \in \mathcal{A}$, over $\mathbb{F}_2$. Let $V^\perp \subset \{0,1\}^n$ denote the subspace orthogonal to V. Since $\mathcal{A}$ is a SUR for $\mathcal{B}$, it follows that for every C_i, there exists a set $A \in \mathcal{A}$ such that $\langle X_{C_i}, X_A \rangle = 1(\mod 2)$ (since c is odd). Therefore, $X_{C_i} \notin V^\perp$, for all $X_{C_i} \in \mathcal{C} = \binom{[n]}{k}$. In other words, $V^\perp$ does not contain any vector consisting of exactly k ones. Moreover, observe that for any $x, y \in V^\perp$, the number of ones in $x + y$ is same as the Hamming distance between x and y. Thus, $V^\perp$ is k-avoiding. Since $\epsilon n < k < (1-\epsilon)n$ and k is even, from Theorem

5.17, it follows that there exists a positive constant $\delta = \delta(\epsilon)$ such that $|V^{\perp}| \leq 2^{n(1-\delta)}$. So, dimension of $V^{\perp}$ is at most $n(1-\delta)$. Therefore, it follows that dimension of V is at least δn. $\square$

Corollary 5.17.1 *$\gamma(n, \frac{n}{2}, r) \geq \delta n$ provided $\frac{n}{2}$ is even and $\frac{r}{2}$ is odd, for some $0 < \delta < 1$.*

Let $\frac{n}{2}$ be even and $\frac{r}{2}$ be odd. From Inequality 5.10, we have $\gamma(n, \frac{n}{2}, r) \in O(n\sqrt{r})$. When r is a constant, using Corollary 5.17.1, this upper bound is asymptotically tight. However, for larger values of r, there can be a large gap (up to $O(\sqrt{n})$ when $r \in \Omega(n)$) between the upper and the lower bound. In what follows, we address the problem for a special case when $r = \frac{n}{2}$ and establish a better upper bound of $\frac{n}{2}$ on $\gamma(n, \frac{n}{2}, \frac{n}{2})$.

Lemma 5.18 *$\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$, where $\frac{n}{2}$ is any even integer.*

Proof. Let $\mathcal{B}$ denote the set of all the bicolorings with equal number of +1's and -1's. Let $A_1 = \{1, 2, \ldots, \frac{n}{2}\}, A_2 = \{2, 3, \ldots, \frac{n}{2}+1\}, \ldots, A_{\frac{n}{2}} = \{\frac{n}{2}, \frac{n}{2}+1, \ldots, n-1\}$. Let $c_i(B) = \langle Y_B, X_{A_i} \rangle$. For any $B \in \mathcal{B}$, it is not hard to see that each $c_i(B)$ is even and $|c_i(B) - c_{i+1}(B)| \in \{0, 2\}$. Since the bicolorings consist of equal number of +1's and -1's, $c_{\frac{n}{2}}(B) \leq -c_1(B) + 2$ if $c_1(B) \geq 0$, and $c_{\frac{n}{2}}(B) \geq -c_1(B) - 2$ if $c_1(B) < 0$. In particular, we have $c_1(B)c_{\frac{n}{2}}(B) \leq 0$. Since $|c_i(B) - c_{i+1}(B)| \in \{0, 2\}$, this implies the existence of an index i such that $c_i(B) = \langle Y_B, X_{A_i} \rangle = 0$. This concludes the proof that $\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. $\square$

From Corollary 5.17.1 and Lemma 5.18, we have the following theorem.

Theorem 5.19 *$\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. Moreover, $\gamma(n, \frac{n}{2}, \frac{n}{2}) \geq \delta n$ if $n/2$ is even and $n/4$ is odd, for some $0 < \delta < 1$.*

5.4 Inapproximability of the SUR problem

Firstly, we establish a hardness result of the hitting set problem for a special family of subsets.

Definition 5.20 *A family $\mathcal{F}$ of subsets of $[n]$ is* complement closed on $[n]$ *if for all $F \in \mathcal{F}$, $[n] \setminus F \in \mathcal{F}$.*

Proposition 5.21 *Let n and m be integers. No deterministic polynomial time algorithm can approximate the hitting set problem for a complement closed family consisting of m distinct subsets of $[n]$ to within a factor of $(1 - \Omega(1))\frac{\ln m}{4}$ of the optimal, unless P=NP.*

Proof. For the sake of contradiction, assume that there exists an algorithm ALG that approximates the hitting set for complement closed families consisting of m sets on $[n]$ to within a factor of $(1 - \Omega(1))\frac{\ln m}{4}$ of the optimal. We obtain a contradiction to this assumption by the following reduction from the general hitting set problem.

Given a pair $(\mathcal{S}', [n])$ as input to the general hitting set problem, we extend the universe to $[n+1]$ by adding the element $n+1$. We construct $\mathcal{S}$ as follows: $\mathcal{S} = \mathcal{S}' \cup \{[n+1] \setminus S | S \in \mathcal{S}'\}$. Note that

$$|\mathcal{S}| \leq 2|\mathcal{S}'| = 2m. \tag{5.11}$$

Let $OPT(\mathcal{S})$ ($OPT(\mathcal{S}')$) denote an optimal solution to the hitting set problem on $\mathcal{S}$ (respectively, $\mathcal{S}'$). Let $ALG(\mathcal{S})$ denote a hitting set outputted by ALG on $\mathcal{S}$ as input.

Observe that

$$|OPT(\mathcal{S}')| \leq |OPT(\mathcal{S})| \leq |OPT(\mathcal{S}')| + 1 \leq 2|OPT(\mathcal{S}')|. \tag{5.12}$$

From our assumption, it follows that $|OPT(\mathcal{S})| \leq |ALG(\mathcal{S})| \leq (1 - \Omega(1))\frac{\ln(2m)}{4}$ $|OPT(\mathcal{S})| < (1 - \Omega(1))\frac{\ln m}{2}|OPT(\mathcal{S})|$. Note that $ALG(\mathcal{S})$ is a valid hitting set for $\mathcal{S}'$. So, $|OPT(\mathcal{S}')| \leq |OPT(\mathcal{S})| \leq |ALG(\mathcal{S})| \leq (1 - \Omega(1))\frac{\ln m}{2}|OPT(\mathcal{S})| < (1 - \Omega(1))\frac{\ln m}{2} \cdot 2|OPT(\mathcal{S}')| = (1 - \Omega(1))\ln m|OPT(\mathcal{S}')|$. Therefore, ALG is a $(1 - \Omega(1))\ln m$ factor approximation algorithm for the general hitting set problem. However, Dinur and Steurer [25] proved that it is impossible to approximate the set cover problem to a factor of $(1 - \Omega(1))\ln n$ of the optimal, unless P=NP. This implies that hitting set problem cannot be approximated to a factor of $(1 - \Omega(1)) \ln m$ of the optimal in polynomial time, unless P=NP. $\square$

We use Proposition 5.21 to establish the following hardness result for the system of unbiased representative problem.

Proof of Theorem 5.5

Statement of Theorem 5.5.

Let n and m be integers and let $r \leq (1 - \Omega(1)) \frac{\ln m}{4}$. Then, no deterministic polynomial time algorithm can approximate the system of unbiased representative problem for a family of m bicolorings on $[n]$ to within a factor $(1 - \Omega(1)) \frac{\ln m}{4r}$ of the optimal when each set chosen in the representative family is required to have its cardinality at most r, unless P=NP.

Proof. We prove Theorem 5.5 by a reduction from an instance of the hitting set problem on complement closed families. Let $\mathcal{S}$ be a complement closed family on $[n]$ of cardinality m. From $\mathcal{S}$, we construct a family $\mathcal{B}$ of bicolorings on $[n]$ in the following way: $\mathcal{B} = \{B | B(+1) = S, B(-1) = [n] \setminus S, S \in \mathcal{S}\}$. For the sake of contradiction, assume that there exists an algorithm ALG that approximates the system of unbiased representative problem for any family of bicolorings on $[n]$ to within a factor f of the optimal, where $1 \leq f \leq (1 - \Omega(1)) \frac{\ln m}{4r}$ and each set in the SUR is required to have its cardinality at most r. Let $OPT_{\text{HIT}}(\mathcal{S})$ ($OPT_{\text{SUR}}(\mathcal{B})$) denote an optimal solution to the hitting set problem (respectively, the system of unbiased representative problem) on $\mathcal{S}$ (respectively, $\mathcal{B}$). Let $ALG(\mathcal{B})$ denote a SUR outputted by ALG with $\mathcal{B}$ as its input. Then, executing ALG on $\mathcal{B}$ as input, we obtain a SUR $\mathcal{A}$ for $\mathcal{B}$ such that (i) $2 \leq |A| \leq r$ for each $A \in \mathcal{A}$, (ii) $|ALG(\mathcal{B})| = |\mathcal{A}| \leq f \cdot |OPT_{\text{SUR}}(\mathcal{B})|$, for some $1 \leq f \leq (1 - \Omega(1)) \frac{\ln m}{4r}$. Let $V = \cup_{A \in \mathcal{A}} A$. It follows that $|V| \leq r|\mathcal{A}|$ and V is a hitting set for $\mathcal{S}$.

From Lemma 5.8, we know that $|OPT_{\text{SUR}}(\mathcal{B})| \le |OPT_{\text{HIT}}(\mathcal{S})| - 1$. Therefore,

$$|OPT_{\text{HIT}}(\mathcal{S})| \le |V| \le r \cdot |ALG(\mathcal{B})| \le r \cdot f \cdot |OPT_{\text{SUR}}(\mathcal{B})| < r \cdot f \cdot |OPT_{\text{HIT}}(\mathcal{S})|.$$

So, ALG is a $(r \cdot f)$-factor approximation algorithm for computing hitting set of $\mathcal{S}$. Since $1 \le f \le (1 - \Omega(1))\frac{\ln m}{4r}$, this is a contradiction to Proposition 5.21. □

Remark 5.22 *Consider the case when the family $\mathcal{B}$ is restricted to a special family of bicolorings, where the number of +1's (or -1's) for each $B \in \mathcal{B}$ is exactly one, i.e. $|B(+1)| = 1$ (or $|B(-1)| = 1$). Then, the problem of system of unbiased representatives reduces to an edge cover problem [72, 53] on a complete graph G, where for each $B \in \mathcal{B}$, a vertex $v_{B(+1)}$ (respectively, $v_{B(-1)}$) is added to $V(G)$. So, this reduction makes the SUR problem polynomial time solvable for such families of bicolorings.*

Chapter 6

Bisection related problems

Let n be any positive integer and $\mathcal{A}$ be a family of subsets of $[n]$. A set B *bisects* another set A if $|A \cap B| \in \{\lceil\frac{|A|}{2}\rceil, \lfloor\frac{|A|}{2}\rfloor\}$. Recall that a family $\mathcal{B}$ of subsets of $[n]$ is called a *bisecting family* for another family $\mathcal{A}$, if for each subset $A \in \mathcal{A}$, there exists a subset $B \in \mathcal{B}$ that bisects A. This problem has been studied in detail in Chapter 3. We consider the following extension of the notion of bisection. A family $\mathcal{A}$ consisting of even subsets of $[n]$ is called *bisection closed* if for each $A, B \in \mathcal{A}$, either A bisects B or B bisects A (or both). In Section 6.1, we study problems related to bisection closed families.

Another interesting problem pertaining to bisecting families is obtaining bisecting families for products of set systems. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two family of subsets of $[n_1]$ and $[n_2]$, respectively, where $n_1, n_2 \in \mathbb{N}$. Let $\beta_{[\pm 1]}(\mathcal{A}_1)$ and $\beta_{[\pm 1]}(\mathcal{A}_2)$ be the minimum cardinality of any *bisecting* family for $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Given $\beta_{[\pm 1]}(\mathcal{A}_1)$ and $\beta_{[\pm 1]}(\mathcal{A}_2)$, $\beta_{[\pm 1]}(\mathcal{A}_1 \Phi \mathcal{A}_2)$ denotes the minimum cardinality of any *bisecting* family for $\mathcal{A}_1 \Phi \mathcal{A}_2$, where Φ represents some product of families $\mathcal{A}_1$ and $\mathcal{A}_2$. In Section 6.2, we study problems related to bisecting families for products of set systems.

6.1 Bisection closed families

Definition 6.1 (Bisection closed families) *A family $\mathcal{A}$ consisting of even subsets of $[n]$ is called **bisection closed** if for each $A, B \in \mathcal{A}$, either A bisects B or B bisects A*

(or both). Let $\vartheta(n)$ ($\vartheta(n,k)$) denote the maximum cardinality of any (respectively, a k-uniform) ***bisection closed*** *family on $[n]$.*

Note that the upper bound on the size of any k-uniform bisection closed family $\mathcal{A}$ on $[n]$ follows from the Fisher's inequality [28, 7]: for every distinct $A, B \in \mathcal{A}$, $|A \cap B| = \frac{k}{2}$. Thus, $\vartheta(n,k) \leq n$. An easy construction of such a family of size $n-1$ is given by a star $K_{1,n-1}$, where $\mathcal{A}$ consists of the edges of the star. A non-trivial construction of a tight example is given by any *Hardamard* matrix of order $4q$, $q \in \mathbb{N}^+$, as follows. Let H denote a $4q \times 4q$ Hadamard matrix where the first row and first column consists of all 1 entries. Since any two rows of the Hadamard matrix are orthogonal, the rows other than the first row of H contain exactly $2q$ 1's and $2q$ -1's. Note that any set of two rows which does not include the first row contain q 1's in common and q -1's in common. Let H' denote the 0–1 matrix of dimension $(4q-1) \times (4q-1)$ obtained from the H by removing (i) the first row and the first column, and, (ii) replacing the -1's by 1's and 1's by 0's. Consider the family $\mathcal{A}$ consisting of subsets represented by the rows of H'. Observe that for any $A, B \in \mathcal{A}$, $|A| = |B| = 2q$ and $|A \cap B| = q$. So, $\mathcal{A}$ is a bisection closed family on $[4q-1]$ with $|\mathcal{A}| = 4q-1$. Therefore, $\vartheta(n,k) = n$.

Since $\mathcal{A}$ can include at most n sets of any particular cardinality (as $\vartheta(n,k) \leq n$), $\vartheta(n) \leq \vartheta(n,k)\{\frac{n}{2} - 1\} + 1 = \frac{n^2}{2} - n + 1$ (since only one set of cardinality n exists). Moreover, if $[n] \in \mathcal{A}$, the only other sized subsets allowed in $\mathcal{A}$ are of size exactly $\frac{n}{2}$. In that case, $|\mathcal{A}|$ cannot exceed $n+1$. Therefore, if $\vartheta(n) > n+1$, any bisection closed family cannot include the set $[n]$. Let A and B be two subsets of $[n]$ with $|A| > \frac{2n}{3}$ and $|B| > \frac{2n}{3}$. It is not hard to see that neither A can bisect B nor B can bisect A. So, in any maximum cardinality bisection closed family, at most one set of cardinality strictly more than $\frac{2n}{3}$ can be present. Thus, $\vartheta(n) \leq \frac{n^2}{3} + 1$.

Lemma 6.2 *Let $\mathcal{A}$ be a bisection closed family of subsets of $[n]$ such that for any pair $A, B \in \mathcal{A}$, B bisects A if and only if $|A| \leq |B|$. Then, $\mathcal{A} \leq n+1$.*

Proof. The lemma follows from the Independence criterion of vectors in a vector space [36]. □

Corollary 6.2.1 *Let $\mathcal{A}$ be a bisection closed family of subsets of $[n]$ let $k_1 < k_2 < \ldots < k_t$ be the different sizes of elements in $\mathcal{A}$, for some $1 \leq t \leq \frac{n}{2}$. If $k_i > 2k_{i-1}$ for all $2 \leq i \leq t$, then $\mathcal{A} \leq n+1$.*

Proof. This follows directly from Lemma 6.2.

6.1.1 Two examples

We have two different constructions of families that are bisection closed and are of cardinality $\frac{3n}{2} - 2$ on $[n]$. Let $\mathcal{B}$ denote the 2-sized sets that contain 1 as the common element, i.e. $\{1,2\}, \{1,3\}, \ldots, \{1,n\}$; and let $\mathcal{C}$ denote 4-sized sets that contain $\{1,2\}$ as the common element, i.e. $\{1,2,3,4\}, \{1,2,5,6\}, \ldots, \{1,2,n-1,n\}$. Let $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$. It is not hard to see that any pair of two element sets in $\mathcal{B}$ are pairwise bisecting since all of them contain 1 as the common element. Similarly, any pair of four-element sets in $\mathcal{C}$ are pairwise bisecting since all of them contain 1 and 2 as the common elements. In order to verify the bisection property between a $B \in \mathcal{B}$ and a $C \in \mathcal{C}$, without loss of generality, consider the set $\{1,2,3,4\}$. The sets $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ bisect $\{1,2,3,4\}$. For any $B \in \mathcal{B} \setminus \{\{1,2\},\{1,3\},\{1,4\}\}$, $\{1,2,3,4\}$ bisects B. Therefore, $\mathcal{A}$ is indeed bisection closed and $|\mathcal{A}| = \frac{3n}{2} - 2$.

The second example of a bisection closed family of cardinality $\frac{3n}{2} - 2$ comes from a special construction of the Hardamard matrices. A recursive Hardamard matrix $H(k)$ of size $2^k \times 2^k$ can be obtained from $H(k-1)$ of size $2^{k-1} \times 2^{k-1}$ as follows

$$H(k) = \begin{bmatrix} H(k-1) & H(k-1) \\ H(k-1) & -H(k-1) \end{bmatrix},$$

where $H(0) = 1$. Now consider the matrix:

$$M(k) = \begin{bmatrix} H(k-1) & H(k-1) \\ H(k-1) & -H(k-1) \\ H(k-1) & -J(k-1) \end{bmatrix}, \text{ where } J(k-1) \text{ denotes the all ones' matrix.}$$

Let $M'(k)$ be the matrix obtained from $M(k)$ by removing the 1st and (2^k+1)th rows and replacing the -1's by 0s. $M'(k)$ is clearly bisection-closed and has cardinality $\frac{3n}{2}-2$, where $n = 2^k$. So, we have constructions of bisection closed families of cardinality $\frac{3n}{2}-2$ whereas the upper bound is $O(n^2)$. In what follows, we improve the upper bound to $O(\frac{n(\log n)^2}{\log\log n})$ using the notion of linear independence over vector spaces.

6.1.2 Upper bounds on $\vartheta(n)$

Let $\mathcal{A}$ be a bisection closed family of subsets of $[n]$ of maximum cardinality. Fix a prime $p > 2$ and partition $\mathcal{A}$ into p parts $\{\mathcal{A}_0, \ldots, \mathcal{A}_{p-1}\}$, where $\mathcal{A}_i = \{A \in \mathcal{A} | |A| = i \mod p\}$.

Estimation of $|\mathcal{A}_i|$ for $i > 0$

Let $\mathcal{A}_i = \{A_1, \ldots, A_m\}$ and let $a_1, \ldots, a_m$ denote their corresponding 0–1 incidence vectors. Construct m polynomials, f_1 to f_m, in the following way.

$$f_j(x) = \langle a_j, x\rangle - \frac{i}{2}, \text{ for } 1 \le j \le m,$$

Note that since $|A_j| \equiv i \pmod p$, $\langle a_j, a_j\rangle \equiv i \pmod p$. Since $p > 2$, $i \not\equiv \frac{i}{2} \pmod p$ unless $i \equiv 0 \pmod p$. So,

$$f_j(x) \begin{cases} \neq & 0, \text{ if } x = a_j \\ = & 0, \text{ otherwise.} \end{cases}$$

So, f_j's are linearly independent in the vector space ${\mathbb{F}_p}^{\{0,1\}^n}$ over $\mathbb{F}_p$ (see [36, Lemma 13.11]). Each f_j is an appropriate linear combination of distinct monomials of degree at most one. Thus, $|\mathcal{A}_i| \le n+1$. However, this technique is useless for estimating $|\mathcal{A}_0|$ since $i \equiv \frac{i}{2} \pmod p$ when $i \equiv 0 \pmod p$.

To overcome this problem, consider the collection $P = \{p_1, \ldots, p_r\}$ of r smallest primes $2 < p_1 < \ldots < p_r$ such that for any $2 \leq |A| \leq n$, there exists a prime $p \in P$ with $p \nmid |A|$. Note that if we repeat the steps done above for each $p \in P$, we obtain the following upper bound.

$$\vartheta(n) \leq (p_1 + \ldots + p_r - r)(n+1) < (rp_r - r)(n+1) = r(p_r - 1)(n+1) < rp_r n. \tag{6.1}$$

To obtain a small cardinality set P of the desired requirement, we choose the minimum r such that $p_1 p_2 \ldots p_r > n$. The product of first t primes is the Primorial function $p_t\#$ and $p_t\# = e^{(1+o(1))t \ln t}$ (see [64]). Since $\frac{p_t\#}{2} = p_1 p_2 \ldots p_r$, setting $e^{(1+o(1))r \ln r - 1} > n$, we get, $r \ln r > \ln n + 1$. Moreover, using the Prime number theorem, the rth prime p_r is at most $2r \ln r$. Note that by setting $r = 2\frac{\ln n}{\ln \ln n}$, we get $\ln n + 1 < r \ln r < 2 \ln n$ for $n \geq 30$. Therefore, from Inequality 6.1, we have the following theorem.

Theorem 6.3 *Let n be an integer more than 30. Then $\vartheta(n) \leq \frac{8n(\ln n)^2}{\ln \ln n}$.*

Remark 6.4 *The bound obtained in Theorem 6.3 can be improved by a constant factor by using a tighter upper bound for p_r.*

6.1.3 Bisection closed families restricted to sets of cardinality more than $\frac{n}{2}$

Let $\mathcal{A}$ be a bisection closed family of maximum cardinality, where each $A \in \mathcal{A}$ has cardinality strictly greater than $\frac{n}{2}$. We use the following lemma that establishes an upper bound on the cardinality of a collection of unit vectors under restrictions on dot products.

Lemma 6.5 *Let $X_1, \ldots, X_m$ be unit vectors in $\mathbb{R}^n$ such that $\langle X_i, X_j \rangle \leq -\gamma$, for some $0 < \gamma < 1$ and $i \neq j$. Then, $m \leq \frac{1}{\gamma} + 1$.*

Proof. Let $u = \sum_{i=1}^{n} X_i$. So, we have, $0 \leq \|u\|^2 = \sum_i \|X_i\|^2 + \sum_{i \neq j} \langle X_i, X_j \rangle \leq m + m(m-1)(-\gamma) = m(1 - \gamma(m-1))$. It follows that $m \leq \frac{1}{\gamma} + 1$. □

When the sets are viewed as vectors in $\{-1, +1\}^n$, the phenomenon of bisection can be realized by the following lemma.

Lemma 6.6 *Let $Y_1, Y_2 \in \{-1, 1\}^n$ be incidence vectors corresponding to sets $A_1, A_2 \subseteq [n]$, where $Y(i) = 1$ if $i \in A$ and $Y(i) = -1$ otherwise. If A_1 bisects A_2, then $\langle Y_1, Y_2 \rangle = n - 2|A_1|$.*

Proof. If A_1 bisects A_2, then

$$
\begin{aligned}
\langle Y_1, Y_2 \rangle =& (n - |A_1| - \frac{|A_2|}{2}) \cdot 1 && \text{(both } Y_1(i), Y_2(i) \text{ are -1)} \\
&+ \frac{|A_2|}{2} \cdot 1 && \text{(both } Y_1(i), Y_2(i) \text{ are 1)} \\
&+ (|A_1| - \frac{|A_2|}{2}) \cdot (-1) && (Y_1(i) \text{ is 1, } Y_2(i) \text{ is -1)} \\
&+ (\frac{|A_2|}{2}) \cdot (-1) && (Y_1(i) \text{ is -1, } Y_2(i) \text{ is 1)} \\
\implies \langle Y_1, Y_2 \rangle =& n - 2|A_1|.
\end{aligned}
$$

□

We have the following theorem.

Theorem 6.7 *Let n be an even integer. Let $\mathcal{A}$ be a bisection closed family of maximum cardinality, where each $A \in \mathcal{A}$ has cardinality strictly greater than $\frac{n}{2}$ and $|A|$ is even. Then $|\mathcal{A}| \leq \frac{n}{2} + 1$.*

Proof. For any $A \in \mathcal{A}$, let $Y_A \in \mathbb{R}^n$ be defined as

$$
Y_A(i) = \begin{cases} \frac{1}{\sqrt{n}}, \text{ if } i \in A \\ -\frac{1}{\sqrt{n}}, \text{ if } i \notin A. \end{cases} \tag{6.2}
$$

In particular, $\|Y_A\|^2 = 1$. So, Y_A is a unit vector corresponding to A.

From Lemma 6.6, we have the following observation regarding the dot products of distinct Y_A and Y_B.

$$\langle Y_A, Y_B \rangle = \begin{cases} \frac{n-2|A|}{n}, & \text{if } A \text{ bisects } B, \\ \frac{n-2|B|}{n}, & \text{if } B \text{ bisects } A. \end{cases} \tag{6.3}$$

Since $|A| > \frac{n}{2}$ and $|B| > \frac{n}{2}$, it follows that $\langle Y_A, Y_B \rangle \le -\frac{2}{n}$. So, using Lemma 6.5, we get, $|\mathcal{A}| \le \frac{n}{2} + 1$. □

6.1.4 Bisection closed families restricted to sets of cardinality $\frac{n}{2} \pm \frac{\sqrt{n}}{2\delta}$

From Equation 6.3, it is clear that in order to use Lemma 6.5 for establishing a linear upper bound for cardinality of a bisection closed family, the sets must be of cardinality strictly more than $\frac{n}{2}$. To obtain linear upper bounds when sets are of cardinality $\frac{n}{2} \pm \frac{\sqrt{n}}{2\delta}$ when $\delta > 1$, we use the following lemma (see [4, 17]).

Lemma 6.8 *[4, 17] Let A be an $m \times m$ real symmetric matrix with $a_{i,i} = 1$ and $|a_{i,j}| \le \epsilon$ for all $i \neq j$. Let $tr(A)$ denote the trace of A, i.e., the sum of the diagonal entries of A. Let $rk(A)$ denote the rank of A. Then,*

$$rk(A) \ge \frac{(tr(A))^2}{tr(A^2)} = \frac{m^2}{m + m(m-1)\epsilon^2}$$

.

Proof. Let $\lambda_1, \ldots, \lambda_m$ denote the eigenvalues of A. Since only $rk(A)$ eigenvalues of A are nonzero, $tr(A) = \sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{rk(A)} \lambda_i = m$. Further, $tr(A^2) = \sum_{i=1}^{m} \lambda_i^2 = \sum_{i=1}^{rk(A)} \lambda_i^2 \le m + m(m-1)\epsilon^2$. Using the Cauchy-Schwartz Inequality, $(\sum_{i=1}^{rk(A)} \lambda_i)^2 \le rk(A) \sum_{i=1}^{rk(A)} \lambda_i^2$. □

We have the following theorem.

Theorem 6.9 *Let n be an even integer and let $\delta > 1$. Let $\mathcal{A}$ be a bisection closed family of maximum cardinality, where each $A \in \mathcal{A}$ has cardinality in the range $[\frac{n}{2} - \frac{\sqrt{n}}{2\delta}, \frac{n}{2} + \frac{\sqrt{n}}{2\delta}]$. and $|A|$ is even. Then, $|\mathcal{A}| \leq \frac{\delta^2}{\delta^2-1}n$.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ denote a bisection closed family where $|A_i| \in [\frac{n}{2} - \frac{\sqrt{n}}{2\delta}, \frac{n}{2} + \frac{\sqrt{n}}{2\delta}]$ and $|A_i|$ is even, for each $1 \leq i \leq m$. With respect to each A_i, let $Y_{A_i} \in \{-1, +1\}^n$ denote the unit vector constructed as in the proof of Theorem 6.7. Let B be the $m \times n$ matrix with $Y_{A_1}, \ldots, Y_{A_m}$ as its rows. Then, from Equation 6.3, it follows that BB^T is an $m \times m$ real symmetric matrix with the diagonal entries being 1 and the absolute value of any other entry being $|\frac{n-2|A|}{n}| \leq \frac{1}{\delta\sqrt{n}}$. From Lemma 6.8, $rk(BB^T) \geq \frac{m}{1+\frac{m-1}{\delta^2 n}} > \frac{m}{1+\frac{m}{\delta^2 n}}$. We know that $rk(BB^T) \leq n$. So, $n \geq m - \frac{m}{\delta^2}$ which completes the proof of the theorem. □

6.2 Hypergraph products

In this section, we study the problem of bisection for products of set systems. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two family of subsets of $[n_1]$ and $[n_2]$, respectively, where $n_1, n_2 \in \mathbb{N}$. Let $\beta_{[\pm 1]}(\mathcal{A}_1)$ and $\beta_{[\pm 1]}(\mathcal{A}_2)$ be the cardinality of any minimum *bisecting* family for $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We consider 3 different product notions and study $\beta_{[\pm 1]}(\mathcal{A}_1 \Phi \mathcal{A}_2)$, where Φ represents any one of the products.

All the product families $\mathcal{A}_1 \Phi \mathcal{A}_2$ have ground set $X = [n_1] \times [n_2]$ where $\times$ denotes the Cartesian product. The product sets differ only on the way the subsets are derived. Let $\mathcal{A} = \mathcal{A}_1 \Phi \mathcal{A}_2$.

6.2.1 Cartesian Products

In the case when Φ is the Cartesian product $\times$, the product set $\mathcal{A}$ is defined as follows.

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 = \{x \times f_2 | x \in [n_1] \text{ and } f_2 \in \mathcal{A}_2\} \cup \{f_1 \times y | f_1 \in \mathcal{A}_1 \text{ and } y \in [n_2]\}.$$

Here, $x \times f_2 = \{(x, a) : a \in f_2\}$ and has cardinality $|f_2|$.

If at least one of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a family of only even subsets, then we can show that $\beta_{[\pm 1]}(\mathcal{A}) \leq \beta_{[\pm 1]}(\mathcal{A}_1) + \beta_{[\pm 1]}(\mathcal{A}_2)$ as follows. Let $\mathcal{B}_1$ be an optimal bisecting family of cardinality t for $\mathcal{A}_1$ and let $\mathcal{B}_2$ be an optimal bisecting family of cardinality r for $\mathcal{A}_2$, where $t, r \in \mathbb{N}$. A bisecting family $\mathcal{B}$ of cardinality $t + r$ for $\mathcal{A}$ can be constructed as follows. $\mathcal{B} = \{A \times [n_2] | A \in \mathcal{B}_1\} \cup \{[n_1] \times B | B \in \mathcal{B}_2\}$.

Example 6.10 *Let* $n_1 = \{1, 2, 3, 4\}$, $n_2 = \{5, 6, 7, 8\}$. *Then,* $X = [n_1] \times [n_2] = \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$. *Let* $\mathcal{A}_1 = \{\{1, 2\}, \{1, 3\}, \{3, 4\}\}$. *Let* $\mathcal{A}_2 = \{\{5, 6, 7\}, \{7, 8\}\}$. *Then,* $B_1 = \{1, 4\}$ *and* $B_2 = \{7\}$ *bisects* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, respectively. From definition, it follows that* $\mathcal{A} = \mathcal{A}_1 \Phi \mathcal{A}_2 = \{\{(1, 5), (1, 6), (1, 7)\}, \{(1, 7), (1, 8)\}, \{(2, 5), (2, 6), (2, 7)\}, \{(2, 7), (2, 8)\}, \{(3, 5), (3, 6), (3, 7)\}, \{(3, 7), (3, 8)\}, \{(4, 5), (4, 6), (4, 7)\}, \{(4, 7), (4, 8)\}, \{(1, 5), (2, 5)\}, \{(1, 5), (3, 5)\}, \{(3, 5), (4, 5)\}, \{(1, 6), (2, 6)\}, \{(1, 6), (3, 6)\}, \{(3, 6), (4, 6)\}, \{(1, 7), (2, 7)\}, \{(1, 7), (3, 7)\}, \{(3, 7), (4, 7)\}, \{(1, 8), (2, 8)\}, \{(1, 8), (3, 8)\}, \{(3, 8), (4, 8)\}\}$. $\mathcal{B} = \{\{(1, 5), (1, 6), (1, 7), (1, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}, \{(1, 7), (2, 7), (3, 7), (4, 7)\}\}$ *is a bisecting family for* $\mathcal{A}$*.*

However, this construction does not work when both $\mathcal{A}_1$ and $\mathcal{A}_2$ includes odd subsets.

We have a non-trivial construction improving the upper bound and we also show that the bound obtained is tight. This construction also works irrespective of whether $\mathcal{A}_1$ and $\mathcal{A}_2$ includes odd subsets or not.

Theorem 6.11 *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be families consisting of subsets of* $[n_1]$ *and* $[n_2]$*, respectively. Then,*

$$\beta_{[\pm 1]}(\mathcal{A}_1 \times \mathcal{A}_2) = \max(\beta_{[\pm 1]}(\mathcal{A}_1), \beta_{[\pm 1]}(\mathcal{A}_2)).$$

Proof. Firstly, we show that $\beta_{[\pm 1]}(\mathcal{A}_1 \times \mathcal{A}_2) \leq \max(\beta_{[\pm 1]}(\mathcal{A}_1), \beta_{[\pm 1]}(\mathcal{A}_2))$. Let $\mathcal{B}_1 = \{B_{11}, \ldots, B_{1t}\}$ be an optimal bisecting family for $\mathcal{A}_1$ and let $\mathcal{B}_2 = \{B_{21}, \ldots, B_{2r}\}$

be an optimal bisecting family for $\mathcal{A}_2$, where $t, r \in \mathbb{N}$. Without loss of generality, assume that $t \geq r$. A bisecting family $\mathcal{B} = \{B_1, \ldots, B_t\}$ of cardinality t for $\mathcal{A}$ can be constructed as follows. For any point $(x, y) \in X$, where $x \in [n_1]$ and $y \in [n_2]$, add (x, y) to B_j if either $x \in B_{1j}$ and $y \notin B_{2j}$ or $x \notin B_{1j}$ and $y \in B_{2j}$. The set B_{2j} is empty for $r < j \leq t$. We claim that $\mathcal{B}$ is indeed a bisecting family for $\mathcal{A}$. Consider an element $A = \{(x, y_1), (x, y_2), \ldots, (x, y_k)\}$ of $\mathcal{A}$. $\{y_1, \ldots, y_k\}$ must be an element of $\mathcal{A}_2$. Let $B_{2j} \in \mathcal{B}_2$ be the subset that bisects $\{y_1, \ldots, y_k\}$. As per the construction of B_j, B_j bisects $A = \{(x, y_1), (x, y_2), \ldots, (x, y_k)\}$ irrespective of whether $x \in B_{1j}$ or not. Similarly, for any $A = \{(x_1, y), (x_2, y), \ldots, (x_k, y)\}$, it is easy to see that A is bisected by $B_j \in \mathcal{B}$ whenever $\{x_1, \ldots, x_k\}$ is bisected by B_{1j}. This concludes the proof of the upper bound.

To see that $\beta_{[\pm 1]}(\mathcal{A}_1 \times \mathcal{A}_2) \geq \max(\beta_{[\pm 1]}(\mathcal{A}_1), \beta_{[\pm 1]}(\mathcal{A}_2))$, observe that for any fixed $x \in [n_1]$ (and $y \in [n_2]$), (i) $X_x = \{x\} \times [n_2]$ (respectively, $X_y = [n_1] \times \{y\}$) is isomorphic to $[n_2]$ (respectively, $[n_1]$), and, (ii) $\mathcal{A}_x = \{\{x\} \times f_2 : f_2 \in \mathcal{A}_2\}$ (respectively, $\mathcal{A}_y = \{f_1 \times \{y\} : f_1 \in \mathcal{A}_1\}$) is isomorphic to $\mathcal{A}_2$ (respectively, $\mathcal{A}_1$). So, $\beta_{[\pm 1]}(\mathcal{A}_x) \geq \beta_{[\pm 1]}(\mathcal{A}_2)$ and $\beta_{[\pm 1]}(\mathcal{A}_y) \geq \beta_{[\pm 1]}(\mathcal{A}_1)$. This establishes the above theorem. $\square$

6.2.2 Square products

In the case when Φ is the Square product $\square$, the product set $\mathcal{A}$ is defined as follows.

$$\mathcal{A} = \mathcal{A}_1 \square \mathcal{A}_2 = \{f_1 \times f_2 | f_1 \in \mathcal{A}_1 \text{ and } f_2 \in \mathcal{A}_2\}.$$

When both the families $\mathcal{A}_1$ and $\mathcal{A}_2$ consists of even subsets of $[n_1]$ and $[n_2]$, respectively, obtaining a small sized bisecting family for $\mathcal{A}_1 \square \mathcal{A}_2$ is easy, as given by the following theorem.

Theorem 6.12 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be families consisting of even sized subsets of $[n_1]$ and*

$[n_2]$, *respectively. Then,*

$$\beta_{[\pm 1]}(\mathcal{A}_1 \square \mathcal{A}_2) = \min(\beta_{[\pm 1]}(\mathcal{A}_1), \beta_{[\pm 1]}(\mathcal{A}_2)).$$

Proof. Without loss of generality, let $t = \beta_{[\pm 1]}(\mathcal{A}_1) \leq \beta_{[\pm 1]}(\mathcal{A}_2)$. Let $\mathcal{B}_1 = \{B_{11}, \ldots, B_{1t}\}$ be an optimal bisecting family for $\mathcal{A}_1$ and let $\mathcal{B}_2 = \{B_{21}, \ldots, B_{2r}\}$ be an optimal bisecting family for $\mathcal{A}_2$, where $t, r \in \mathbb{N}$. A bisecting family $\mathcal{B} = \{B_1, \ldots, B_t\}$ of cardinality t for $\mathcal{A}$ can be constructed as follows. For any point $(x, y) \in X$, where $x \in [n_1]$ and $y \in [n_2]$, add (x, y) to B_j if either $x \in B_{1j}$ and $y \notin B_{2j}$ or $x \notin B_{1j}$ and $y \in B_{2j}$. B_j is considered empty for $r < j \leq t$. We claim that $\mathcal{B}$ is indeed a bisecting family for $\mathcal{A}$. Observe that for any $f_1 \times f_2$ where $f_1 \in \mathcal{A}_1$ and $f_2 \in \mathcal{A}_2\}$, $f_1 \times f_2$ is bisected by B_i if f_1 is bisected by B_{1i}. □

Example 6.13 *In Example 6.10, choosing the same values for* n_1, n_2, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, *and* $\mathcal{B}_2$, *it follows that* $\mathcal{A} = \mathcal{A}_1 \square \mathcal{A}_2 = \{\{(1,5),(1,6),(1,7),(2,5),(2,6),(2,7)\},\{(1,7),(1,8),(2,7),(2,8)\},\{(1,5),(1,6),(1,7),(3,5),(3,6),(3,7)\},\{(1,7),(1,8),(3,7),(3,8)\},\{(3,5),(3,6),(3,7),(4,5),(4,6),(4,7)\},\{(3,7),(3,8),(4,7),(4,8)\}\}$. *Then,* $\mathcal{B} = \{\{(2,7),(3,7),(1,5),(4,5),(1,6),(4,6),(1,8),(4,8)\}\}$ *is a bisecting family for* $\mathcal{A}$.

However, this construction does not work when at least one of $\mathcal{A}_1$ and $\mathcal{A}_2$ includes odd subsets. If at least one of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a family of only even subsets, the construction given in the proof of Theorem 6.12 yields the following bound.

Theorem 6.14 *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be families consisting of subsets of* $[n_1]$ *and* $[n_2]$, *respectively. Moreover, let each element of the family* $\mathcal{A}_1$ *be even sized. Then,* $\beta_{[\pm 1]}(\mathcal{A}_1 \square \mathcal{A}_2) \leq \beta_{[\pm 1]}(\mathcal{A}_1)$.

Proof. Let $t = \beta_{[\pm 1]}(\mathcal{A}_1)$. Let $\mathcal{B}_1 = \{B_{11}, \ldots, B_{1t}\}$ be an optimal bisecting family for $\mathcal{A}_1$. Consider the family $\mathcal{B} = \{B_1 \times [n_2], \ldots, B_t \times [n_2]\}$. We claim that $\mathcal{B}$ is a bisecting family for $\mathcal{A} = \mathcal{A}_1 \square \mathcal{A}_2$. Note that for any $f_1 \in \mathcal{A}_1$ and $f_2 \in \mathcal{A}_2$, f_1 is bisected by some $B_{1j} \in \mathcal{B}_1$, $1 \leq j \leq t$. So, the set $f_1 \times f_2$ must also get bisected by $F'_j \times [n_2]$. However, this construction does not work when both $\mathcal{A}_1$ and $\mathcal{A}_2$ includes odd subsets, as demonstrated in the following example.

Example 6.15 *Let* $n_1 = \{1,2,3,4,5\}$, $n_2 = \{6,7,8,9,10,11,12,13\}$. *Let* $\mathcal{A}_1 = \{\{1,2,3,4,5\}\}$. *Let* $\mathcal{A}_2 = \{\{6,7,8\},\{9,10,11,12,13\},\ldots\}$ *such that* $\mathcal{B}_2 = \{\{6\},\{9,10\}\}$ *is an optimal bisecting family for* $\mathcal{A}_2$. *Then,* $B_1 = \{1,2,3\}$ *bisects* $\mathcal{A}_1$, *and* $\mathcal{B}_2 = \{\{6\},\{9,10\}\}$ *bisects* $\mathcal{A}_2$. *Let* $\mathcal{B} = \{\{\{1,2,3\}\times\{9,10,11,12,13\}\cup\{4,5\}\times\{6,7,8\}\},\{[n_1]\times\{9,10\}\}\}$. *The hyperedge* $\{\{1,2,3,4,5\}\times\{9,10,11,12,13\}\}$ *is not bisected by* $\mathcal{B}$.

In this case, it is not hard to prove that $\beta_{[\pm 1]}(\mathcal{A}_1 \Box \mathcal{A}_2) \leq \beta_{[\pm 1]}(\mathcal{A}_1) \cdot \beta_{[\pm 1]}(\mathcal{A}_2)$, but this upper bound may not be tight.

6.2.3 Strong products

In the case when Φ is the Strong product $\blacksquare$, the product set $\mathcal{A}$ is defined as follows.

$$\mathcal{A} = \{\mathcal{A}_1 \times \mathcal{A}_2\} \cup \{\mathcal{A}_1 \Box \mathcal{A}_2\}.$$

We have the following result that follows from the discussion above.

1. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ consists of even subsets: From Theorems 6.11 and 6.12, it follows that $\beta_{[\pm 1]}(\mathcal{A}_1 \blacksquare \mathcal{A}_2) = \max(\beta_{[\pm 1]}(\mathcal{A}_1), \beta_{[\pm 1]}(\mathcal{A}_2))$.

2. At least one of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a family of only even subsets: From Theorems 6.11 and 6.14, it follows that $\beta_{[\pm 1]}(\mathcal{A}_1 \blacksquare \mathcal{A}_2) \leq 2\max(\beta_{[\pm 1]}(\mathcal{A}_1), \beta_{[\pm 1]}(\mathcal{A}_2))$.

Chapter 7

Conclusion

In this thesis we have proposed and studied a few extremal two-family problems where we are given one family of sets and are required to optimize the cardinality of another family, satisfying certain constraints with respect to the given family. In the process of studying the cardinalities of such extremal families, we established exact solutions in a few cases and asymptotically tight bounds in some cases. The first three problems are motivated from applications in product testing, especially testing of drugs, whereas the last problem has motivation from a purely combinatorial problem of L-intersecting families. In each of these problems, some questions remain open for further investigation as listed below. We believe some of these open problems are hard while a few others seem more tractable - solutions for these problems may provide new insights regarding the harder open problems.

In Section 3.1.3, we have seen that $\beta_D(n,k)$ is not monotone with k in general. However, it is possible that $\beta_D(n,k)$ is monotone with k in certain ranges, say when $k \leq \frac{n}{2}$. In Section 3.3.2, we established the lower bound of $\frac{n-i+1}{2}$ for $\beta_i(n)$. However, the best upper bound we have for this case is just $n-i+1$. So, there is a gap between the lower and upper bounds for $\beta_i(n)$. These two questions seem tractable for the problem of bisecting families; however, the following question of inapproximability of $\beta_D(E)$ for an n-vertex k-uniform hypergraph $G(V,E)$ seems harder. From the discussions in Chapter 3, it follows that $\beta_{[\pm(k-1)]}(E)$ is $\lceil \log \chi(G) \rceil$ for any k-uniform hypergraph

$G(V,E)$. We know that for any fixed k, it is impossible to approximate the chromatic number of k-uniform hypergraphs on n vertices within a factor of $n^{1-\epsilon}$ for any fixed $\epsilon > 0$, unless $NP \subseteq ZPP$ (see Krivelevich and Sudakov [41], Feige and Killian [27]). Therefore, under the assumption $NP \not\subseteq ZPP$, no polynomial time algorithm can approximate $\beta_{[\pm(k-1)]}(E)$ for an n-vertex k-uniform hypergraph $G(V,E)$ within an additive approximation factor of $(1-\epsilon)\log n - 1$, for any fixed $\epsilon > 0$ and for any fixed k. However, we believe that approximating $\beta_D(E)$ is significantly harder. It seems plausible that approximating $\beta_D(E)$ for an n-vertex k-uniform hypergraph within an additive factor of δn is NP hard, for any fixed $\epsilon > 0$, for some fixed $\delta > 0$ and for any fixed k.

We note that if $d = O(1)$, then Theorem 4.4 asserts that $\beta^d(n) = \theta(n^2)$. However, the corresponding coefficients are not the same: the lower bound has the coefficient $\frac{2}{d^2}$ whereas the upper bound has the coefficient $\frac{2}{(d-1)^2}$. It would be interesting to determine the exact coefficient in this case. Moreover, when d is even and $d \in \Omega(n^{0.5+\epsilon})$, for any $0 < \epsilon \leq 0.5$, we have an upper bound of $O(n)$ on $\beta^d(n)$; the lower bound for this case is $o(n)$. We believe that $\beta^d(n)$ is more close to the upper bound. Though the linear algebraic techniques does not seem to yield tight lower bounds, it may be possible to obtain better lower bounds with some combinatorial arguments.

The constructions establishing tight upper bounds for $\gamma(n, [1, \frac{n}{2}], [2, n])$ in Theorem 5.1 involve two sized sets in the SUR. We note that two sized sets are indispensable in this case due to the presence of bicolorings consisting of exactly one +1 and $n-1$ -1's. However, if the underlying set of bicolorings does not include such biased bicolorings (i.e. say $\gamma(n, [t, \frac{n}{2}], [2, n])$ for some $1 < t \leq \frac{n}{2}$), it is possible to obtain SUR's avoiding two sized sets. Study of $\gamma(n, [t, \frac{n}{2}], [2, n])$ for any $1 < t \leq \frac{n}{2}$ remains open. The restricted SUR problem, where each bicoloring is restricted to have exactly k +1's and each set in the SUR is required to be of cardinality exactly r, can be modelled as a covering problem that enabled us to establish $\frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}} \leq \gamma(n,k,r) \leq \frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}}\left(1 + 0.7r + \ln(\binom{n-r}{k-\frac{r}{2}})\right)$. In Theorem 5.3, under even tighter conditions (i.e., $k \leq \log_4 \log_4(n^{0.5-\epsilon})$ and $r = 2k$), we establish asymptotically tight bounds for $\gamma(n, k, 2k)$ using the 'Rödl nibble method'. It seems plausible that a more prudent selection of nibbles may improve the upper bounds for $\gamma(n, k, r)$ under slightly relaxed conditions. In Section 5.4, we establish that no deterministic polynomial time algorithm can approximate the system of unbiased representative problem for a family of bicolorings on $[n]$ to within a factor $(1 - \Omega(1))\frac{\ln n}{2.34r}$ of the optimal when each set chosen in the representative family is required to have its cardinality at most r, unless P=NP. Improving this hardness of approximation of the system of unbiased representative problem for a family of bicolorings on $[n]$ remains open.

Bisection closed families are a natural generalization of the notion of bisection. In Theorem 6.3, we show that $\vartheta(n)$ is at most $O(n\frac{\log^2 n}{\log\log n})$. However, we suspect that the actual value of $\vartheta(n)$ is more close to $O(n)$. We have two examples of such families of cardinality $\frac{3n}{2} - 2$. In special cases when a bisection closed family is restricted to (i) sets of cardinality more than $\frac{n}{2}$ or (ii) sets of cardinality $\frac{n}{2} \pm \frac{\sqrt{n}}{2\delta}$, we establish that the cardinality of the bisection closed family is $O(n)$. The techniques used in the special cases does not directly extend to the general case. We believe that the lower bound is more close to the actual value of $\vartheta(n)$ and improving the general upper bound for $\vartheta(n)$ is open.

Bibliography

[1] Martin Aigner, Günter M Ziegler, and Alfio Quarteroni. *Proofs from the Book*, volume 274. Springer, 2010.

[2] N. Alon and Z. Füredi. Covering the cube by affine hyperplanes. *European Journal of Combinatorics*, 14(2):79–83, 1993. URL: `https://www.scopus.com/inward/record.uri?eid=2-s2.0-38249006949&partnerID=40&md5=c606f9cd3dcf800edff600fc584f97f9`, `doi:10.1006/eujc.1993.1011`.

[3] Noga Alon. Combinatorial nullstellensatz. *Combinatorics, Probability and Computing*, 8(1-2):7–29, 1999. URL: `https://www.cambridge.org/core/article/combinatorial-nullstellensatz/E32C7ED059D6B92C6C67E18CC716872A`.

[4] Noga Alon. Perturbed identity matrices have high rank: Proof and applications. *Combinatorics, Probability and Computing*, 18(1-2):3–15, 2009.

[5] Noga Alon, Béla Bollobás, Jeong Han Kim, and Van H. Vu. Economical covers with geometric applications. *Proceedings of the London Mathematical Society*, 86(2):273, 2003. URL: `+http://dx.doi.org/10.1112/S0024611502013886`, `arXiv:/oup/backfile/Content_public/Journal/plms/86/2/10.1112/S0024611502013886/2/86-2-273.pdf`, `doi:10.1112/S0024611502013886`.

[6] Noga Alon and Joel H. Spencer. *The probabilistic method.* John Wiley & Sons, 2000.

[7] László Babai and Péter Frankl. *Linear Algebra Methods in Combinatorics: With Applications to Geometry and Computer Science*. Department of Computer Science, univ. of Chicag, 1992.

[8] Nikhil Bansal. Constructive algorithms for discrepancy minimization. In *51th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2010, October 23-26, 2010, Las Vegas, Nevada, USA*, pages 3–10, 2010. URL: `http://dx.doi.org/10.1109/FOCS.2010.7`, `doi:10.1109/FOCS.2010.7`.

[9] Nikhil Bansal and Joel Spencer. Deterministic discrepancy minimization. *Algorithmica*, 67(4):451–471, 2013. URL: `http://dx.doi.org/10.1007/s00453-012-9728-1`, `doi:10.1007/s00453-012-9728-1`.

[10] Manu Basavaraju, Mathew C Francis, M. S. Ramanujan, and Saket Saurabh. Partially polynomial kernels for set cover and test cover. *SIAM Journal on Discrete Mathematics*, 30(3):1401–1423, 2016.

[11] József Beck. Flat polynomials on the unit circle–note on a problem of littlewood. *Bull. London Math. Soc*, 23(3):269–277, 1991.

[12] József Beck and Vera T. Sós. Handbook of combinatorics (vol. 2). chapter Discrepancy Theory, pages 1405–1446. MIT Press, Cambridge, MA, USA, 1995. URL: `http://dl.acm.org/citation.cfm?id=233228.233235`.

[13] Claude Berge. *Graphs and hypergraphs*, volume 7. North-Holland publishing company Amsterdam, 1973.

[14] Béla Bollobás. *Extremal graph theory*. Dover Publications, INC, Mineola, New York, 2004.

[15] L. E. J. Brouwer. Über abbildung von mannigfaltigkeiten. *Mathematische Annalen*, 71(1):97–115, 1911. URL: `http://dx.doi.org/10.1007/BF01456931`, `doi:10.1007/BF01456931`.

[16] Bernard Chazelle. *The Discrepancy Method.* Cambridge University Press, 2000. Cambridge Books Online. URL: http://dx.doi.org/10.1017/CBO9780511626371.

[17] Bruno Codenotti, Pavel Pudlák, and Giovanni Resta. Some structural properties of low-rank matrices related to computational complexity. *Theoretical Computer Science*, 235(1):89 – 107, 2000. URL: http://www.sciencedirect.com/science/article/pii/S0304397599001851, doi:http://dx.doi.org/10.1016/S0304-3975(99)00185-1.

[18] Edward G Coffman Jr, Michael R Garey, and David S Johnson. Approximation algorithms for bin packing: A survey. In *Approximation algorithms for NP-hard problems*, pages 46–93. PWS Publishing Co., 1996.

[19] Robert Crowston, Gregory Gutin, Mark Jones, Saket Saurabh, and Anders Yeo. Parameterized study of the test cover problem. In *International Symposium on Mathematical Foundations of Computer Science*, pages 283–295. Springer, 2012.

[20] Koen M. J. De Bontridder, B. J. Lageweg, Jan K. Lenstra, James B. Orlin, and Leen Stougie. Branch-and-bound algorithms for the test cover problem. In *European Symposium on Algorithms*, pages 223–233. Springer, 2002.

[21] Koen M.J. De Bontridder, Bjarni V. Halldórsson, Magnús M. Halldórsson, Cor A.J. Hurkens, Jan Karel Lenstra, R. Ravi, and Leen Stougie. Approximation algorithms for the test cover problem. *Mathematical Programming*, 98(1):477–491, 2003.

[22] Van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wisk*, 15(2):212–216, 1927.

[23] Pierre A. Devijver and Josef Kittler. *Pattern recognition: A statistical approach*, volume 761. Prentice-Hall London, 1982.

[24] T. J. Dickson. On a problem concerning separating systems of a finite set. *J. Combin. Theory*, 7:191–196, 1969.

[25] Irit Dinur and David Steurer. Analytical approach to parallel repetition. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 624–633. ACM, 2014.

[26] P. Erdós and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. *Colloq. Math. Soc. János Bolya*, 10:609–627, 1975.

[27] Uriel Feige and Joe Kilian. Zero knowledge and the chromatic number. *Journal of Computer and System Sciences*, 57(2):187 – 199, 1998. URL: `http://www.sciencedirect.com/science/article/pii/S0022000098915872`, doi: `http://dx.doi.org/10.1006/jcss.1998.1587`.

[28] Ronald Aylmer Fisher. An examination of the different possible solutions of a problem in incomplete blocks. *Annals of Human Genetics*, 10(1):52–75, 1940.

[29] Peter Frankl and Vojtěch Rödl. Forbidden intersections. *Trans. Amer. Math. Soc.*, 300(1):259–286, 1987. URL: `http://dx.doi.org/10.2307/2000598`, doi: `10.2307/2000598`.

[30] Michael R Garey and David S Johnson. Computers and intractability: a guide to the theory of np-completeness. 1979. *San Francisco, LA: Freeman*, 58, 1979.

[31] Jacques Hadamard. Note sur quelques applications de l'indice de kronecker. In *Introduction à la théorie des fonctions d'une variable*, volume 2, pages 437–477. 2nd edition, 1910.

[32] Bjarni V. Halldórsson, Magnús M. Halldórsson, and R. Ravi. On the approximability of the minimum test collection problem. In *European Symposium on Algorithms*, pages 158–169. Springer, 2001.

[33] David Haussler and Emo Welzl. ϵ-nets and simplex range queries. *Discrete & Computational Geometry*, 2(2):127–151, 1987.

[34] Nils Hebbinghaus and Anand Srivastav. Multicolor discrepancy of arithmetic structures. In *A Panorama of Discrepancy Theory*, pages 319–424. Springer, 2014.

[35] Rebecca Hoberg and Thomas Rothvoss. A logarithmic additive integrality gap for bin packing. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2616–2625. SIAM, 2017.

[36] Stasys Jukna. *Extremal combinatorics: with applications in computer science*. Springer Science & Business Media, 2011.

[37] Narendra Karmarkar and Richard M Karp. An efficient approximation scheme for the one-dimensional bin-packing problem. In *Foundations of Computer Science, 1982. SFCS'08. 23rd Annual Symposium on*, pages 312–320. IEEE, 1982.

[38] G. Katona. On separating systems of a finite set. *J. Combin. Theory I*, pages 174–194, 1966.

[39] Peter Keevash and Eoin Long. Frankl-rödl-type theorems for codes and permutations. *Transactions of the American Mathematical Society*, 369(2):1147–1162, 2017.

[40] János Komlós, János Pach, and Gerhard Woeginger. Almost tight bounds for ϵ-nets. *Discrete & Computational Geometry*, 7(2):163–173, 1992. URL: `http://dx.doi.org/10.1007/BF02187833`, `doi:10.1007/BF02187833`.

[41] Michael Krivelevich and Benny Sudakov. Approximate coloring of uniform hypergraphs. *Journal of Algorithms*, 49(1):2–12, 2003.

[42] S. P. Lapage, Shoshana Bascomb, W. R. Willcox, and M. A. Curtis. Identification of bacteria by computer: general aspects and perspectives. *Microbiology*, 77(2):273–290, 1973.

[43] Nathan Linial and Jaikumar Radhakrishnan. Essential covers of the cube by hyperplanes. *Journal of Combinatorial Theory, Series A*, 109(2):331 – 338, 2005. URL: `http://www.sciencedirect.com/science/article/pii/S0097316504001311`, `doi:http://dx.doi.org/10.1016/j.jcta.2004.07.012`.

[44] L. Lovász. On the ratio of optimal integral and fractional covers. *Discrete Mathematics*, 13(4):383 – 390, 1975. URL: `http://www.sciencedirect.com/science/`

article/pii/0012365X75900588, doi:http://dx.doi.org/10.1016/0012-365X(75)90058-8.

[45] László Lovász. Coverings and colorings of hypergraphs. In *Proc. 4th Southeastern Conf. on Combinatorics, Graph Theory, and Computing*, pages 3–12. Utilitas Mathematica Publishing Winnipeg, 1973.

[46] László Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25(3):319–324, 1978.

[47] Shachar Lovett and Raghu Meka. Constructive discrepancy minimization by walking on the edges. In *Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on*, pages 61–67. IEEE, 2012.

[48] Jiří Matoušek. Combinatorial discrepancy. In *Geometric Discrepancy*, volume 18 of *Algorithms and Combinatorics*, pages 101–135. Springer Berlin Heidelberg, 1999. URL: http://dx.doi.org/10.1007/978-3-642-03942-3_4, doi:10.1007/978-3-642-03942-3_4.

[49] Marvin Minsky and Seymour Papert. Perceptron: an introduction to computational geometry. *The MIT Press, Cambridge, expanded edition*, 19(88):2, 1969.

[50] Bernard M. E. Moret and Henry D. Shapiro. On minimizing a set of tests. *SIAM Journal on Scientific and Statistical Computing*, 6(4):983–1003, 1985.

[51] Robin A. Moser and Gábor Tardos. A constructive proof of the general Lovász Local Lemma. *J. ACM*, 57(2):11:1–11:15, February 2010. URL: http://doi.acm.org/10.1145/1667053.1667060, doi:10.1145/1667053.1667060.

[52] Rajeev Motwani and Prabhakar Raghavan. *Randomized Algorithms*, pages 115–118. Cambridge University Press, New York, NY, USA, 1995.

[53] Katta G Murty and Clovis Perin. A 1-matching blossom-type algorithm for edge covering problems. *Networks*, 12(4):379–391, 1982.

[54] John E Olson and Joel H Spencer. Balancing families of sets. *Journal of Combinatorial Theory, Series A*, 25(1):29–37, 1978.

[55] János Pach and Pankaj K Agarwal. *Combinatorial geometry*, volume 37. John Wiley & Sons, 1995.

[56] R. Paturi and M. E. Saks. On threshold circuits for parity. In *31st Annual Symposium on Foundations of Computer Science, 1990. Proceedings.*, pages 397–404 vol.1, Oct 1990. doi:10.1109/FSCS.1990.89559.

[57] R. W. Payne and D. A. Preece. Identification keys and diagnostic tables: A review. *Journal of the Royal Statistical Society. Series A (General)*, pages 253–292, 1980.

[58] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 2(1):264–286, 1930.

[59] A. Rényi. On random generating elements of a finite boolean algebra. *Acta Sci. Math. (Szeged)*, 22(1-2):75–81, 1961.

[60] Emily Riehl and E. Graham Evans Jr. On the intersections of polynomials and the Cayley–Bacharach theorem. *Journal of Pure and Applied Algebra*, 183(1–3):293 – 298, 2003. URL: http://www.sciencedirect.com/science/article/pii/S002240490300032X, doi:http://dx.doi.org/10.1016/S0022-4049(03)00032-X.

[61] Vojtěch Rödl. On a packing and covering problem. *European Journal of Combinatorics*, 6(1):69 – 78, 1985. URL: http://www.sciencedirect.com/science/article/pii/S0195669885800238, doi:http://dx.doi.org/10.1016/S0195-6698(85)80023-8.

[62] K. F. Roth. Irregularities of sequences relative to arithmetic progressions, iv. *Periodica Mathematica Hungarica*, 2(1-4):301–326, 1972.

[63] Thomas Rothvoß. Approximating bin packing within o (log opt* log log opt) bins. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 20–29. IEEE, 2013.

[64] Sebastian Martin Ruiz. 81.27 a result on prime numbers. *The Mathematical Gazette*, 81(491):269–270, 1997. URL: http://www.jstor.org/stable/3619207.

[65] Michael E. Saks. Slicing the hypercube. In *Surveys in Combinatorics, 1993*, pages 211–256. Cambridge University Press, 1993. URL: http://dx.doi.org/10.1017/CBO9780511662089.009.

[66] David Saxton. Essential positive covers of the cube. *Journal of Combinatorial Theory, Series A*, 120(5):971 – 975, 2013. URL: http://www.sciencedirect.com/science/article/pii/S0097316513000277, doi:http://dx.doi.org/10.1016/j.jcta.2013.02.002.

[67] Victor Shoup. *A computational introduction to number theory and algebra*. Cambridge university press, 2009.

[68] J. Spencer. Minimal completely separating systems. *J. Combin. Theory*, 8:446–447, 1970.

[69] Joel Spencer. Six standard deviations suffice. *Trans. Amer. Math. Soc.*, 289(2):679–706, 1985. URL: http://dx.doi.org/10.2307/2000258, doi:10.2307/2000258.

[70] S. K. Stein. Two combinatorial covering theorems. *Journal of Combinatorial Theory, Series A*, 16(3):391 – 397, 1974. URL: http://www.sciencedirect.com/science/article/pii/0097316574900624, doi:http://dx.doi.org/10.1016/0097-3165(74)90062-4.

[71] I. Wegener. On separating systems whose elements are sets of at most k elements. *Discrete Math.*, 28:219–222, 1979.

[72] Lee J White. Minimum covers of fixed cardinality in weighted graphs. *SIAM Journal on Applied Mathematics*, 21(1):104–113, 1971.

[73] W. R. Willcox and S. P. Lapage. Automatic construction of diagnostic tables. *The Computer Journal*, 15(3):263–267, 1972.

[74] Richard M. Wilson. An existence theory for pairwise balanced designs. I. Composition theorems and morphisms. *J. Combinatorial Theory Ser. A*, 13:220–245, 1972.

[75] Richard M. Wilson. An existence theory for pairwise balanced designs. II. The structure of PBD-closed sets and the existence conjectures. *J. Combinatorial Theory Ser. A*, 13:246–273, 1972.

[76] Richard M. Wilson. An existence theory for pairwise balanced designs. III. Proof of the existence conjectures. *J. Combinatorial Theory Ser. A*, 18:71–79, 1975.

[77] Alex Wright. Sperner's lemma and brouwer's fixed point theorem. 2005. URL: `http://web.stanford.edu/~amwright/BFPT.pdf`.