Bisecting families and related problems

Thesis defence presentation of

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September 07, 2017

Organization of the Thesis

- Bisecting and D-secting families for hypergraphs

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- Bisection closed families

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- Test cover problem (Garey and Johnson, 1979; Moret and Shapiro, 1985; De Bontridder et al., 2002; Crowston et al., 2012; Basavaraju et al., 2016; Payne and Preece, 1980; Willcox and Lapage, 1972; Lapage et al., 1973; Devijver and Kittler, 1982).

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- Covering Hamming cube with affine hyperplanes (Alon and Füredi, 1993; Linial and Radhakrishnan, 2005; Saxton, 2013; Saks, 1993).

Chapter 3
Bisecting and D-secting families for hypergraphs

B bisects A

Definition 2.1

Let $A, B \subseteq [n]$. Then, B **bisects** A if $|A \cap B| \in \{\lfloor \frac{|A|}{2} \rfloor, \lceil \frac{|A|}{2} \rceil\}$.

Example 2.2

Let $n = 10$, $A = \{1, 2, 3, 6, 7, 8\}$, $C = \{4, 5, 6, 7\}$,
$B = \{1, 3, 5, 8, 10\}$. Then, $|B \cap A| = 3 = \frac{|A|}{2}$, $|B \cap C| = 1 \neq \frac{|C|}{2}$.

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$X_A = \{1, 1, 1, 0, 0, 1, 1, 1, 0, 0\}$, $X_C = \{0, 0, 0, 1, 1, 1, 1, 0, 0, 0\}$,
$Y_B = \{1, -1, 1, -1, 1, -1, -1, 1, -1, 1\}$.

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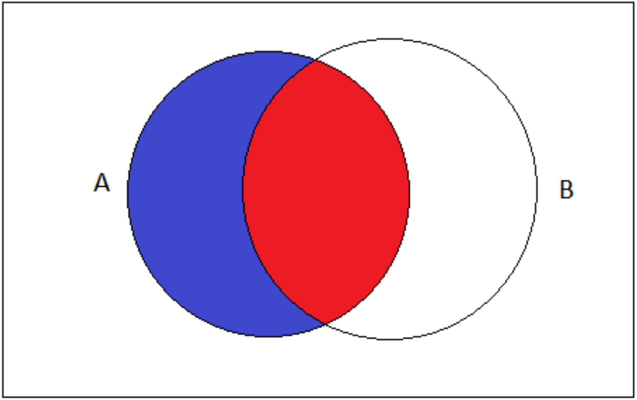
$X_A = \{1, 1, 1, 0, 0, 1, 1, 1, 0, 0\}$, $X_C = \{0, 0, 0, 1, 1, 1, 1, 0, 0, 0\}$,
$Y_B = \{1, -1, 1, -1, 1, -1, -1, 1, -1, 1\}$.
$\langle X_A, Y_B \rangle = 0$, $\langle X_C, Y_B \rangle = -2 \neq 0$.

Equivalent definition



Definition 2.3 (Equivalent)

Let $D = \{-1, 0, 1\}$. B **bisects** A if $|A \cap B| - |A \cap ([n] \setminus B)| \in D$.

B D-sects A - generalizing Definition 2.3

Let $[\pm i]$ denote $\{-i, \ldots, 0, \ldots, i\}$.

Definition 2.4

Let $D \subseteq [\pm n]$. Then, B D-**sects** A if $|A \cap B| - |A \cap ([n] \setminus B)| \in D$ ($\langle X_A, Y_B \rangle \in D$).

Example 2.5

Let $n = 10$, $A = \{1, 2, 3, 6, 7, 8\}$, $C = \{4, 5, 6, 7\}$,
$B = \{1, 3, 5, 8, 10\}$. Then,
$|A \cap B| - |A \cap ([n] \setminus B)| = \langle X_A, Y_B \rangle = 0 \in D$.
$|C \cap B| - |C \cap ([n] \setminus B)| = \langle X_C, Y_B \rangle = -2 \in D$. Therefore, B D-sects both A and C.

$\mathcal{B}$ bisects $\mathcal{A}$

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Definition 2.6

$\mathcal{B}$ is a **bisecting family** for $\mathcal{A}$ if for every $A \in \mathcal{A}$ there exists an $B \in \mathcal{B}$ such that B bisects A.

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Example 2.7

Let $\mathcal{A} = \{\{1,7,3\},\{1,4,5,6\},\{2,3,6,1\},\{2,4,7,8\}\}$. Then, $\mathcal{B} = \{\{1,4\},\{2,6,8\}\}$ bisects $\mathcal{F}$. Another family $\mathcal{B}' = \{\{1,8,2,5\}\}$ of smaller cardinality also bisects $\mathcal{A}$.

$\mathcal{B}$ D-sects $\mathcal{A}$

Definition 2.8

Let $D \subseteq [\pm n]$. $\mathcal{B}$ is a D-**secting family** for $\mathcal{A}$ if for every $A_i \in \mathcal{A}$ there exists an $B_j \in \mathcal{B}$ such that B_j D-sects A_i.

A **bisecting family** is a D-**secting family**, where $D = [\pm 1]$.

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Notations:
$\beta_D(\mathcal{A})$: min. cardinality of a family $\mathcal{B}$ that D-sects $\mathcal{A}$.

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$\beta_D(\mathcal{A})$: min. cardinality of a family $\mathcal{B}$ that D-sects $\mathcal{A}$.
$\beta_D(n)$: max. of $\beta_D(\mathcal{A})$ over all families $\mathcal{A} \subseteq 2^{[n]}$.

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$\beta_D(n, k)$: max. of $\beta_D(\mathcal{A})$ over all families $\mathcal{A} \subseteq \binom{[n]}{k}$.

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$\beta_D(n,k)$: max. of $\beta_D(\mathcal{A})$ over all families $\mathcal{A} \subseteq \binom{[n]}{k}$.
$\beta_{[\pm i]}(\mathcal{A})$: $\beta_D(\mathcal{A})$, when $D = [\pm i]$.

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- $\mathcal{A} \to \mathcal{B}$: $\forall A \in \mathcal{A}$, $\exists B \in \mathcal{B}$ - $\langle X_A, Y_B \rangle \in D$: $\mathcal{B}$ is a D-secting family for $\mathcal{A}$.

Results

$$[\pm i] = \{-i, -i+1, \dots, 0, \dots, i\}.$$

Theorem 2.9

$\beta_{[\pm i]}(n) = \lceil \frac{n}{2i} \rceil$, $n \in \mathbb{N}$, $i \in [n]$. Proof.

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The Chernoff's bound gives

Theorem 2.10

Let $\mathcal{A}$ be a family of subsets of $[n]$ and let $m = |\mathcal{A}|$. Let $i \geq \sqrt{\frac{3n\ln(2m)}{t}}$ and $t \leq \frac{1}{2}\log m$. Then, $\beta_{[\pm i]}(\mathcal{A}) \leq t$. ▸ Proof.

Bisecting k-uniform families

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For a family $\mathcal{F}$ consisting of k-sized subsets of $[n]$ and dependency d, $\beta_{[\pm 1]}(\mathcal{F}) \in O(\sqrt{k} \log d)$.

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Lemma 2.12

$$\beta_{[\pm 1]}(n, k) \geq \begin{cases} \log(n - k + 2), \text{ when } k \text{ is even and } \frac{k}{2} \text{ is odd,} \\ \lceil (\log \lceil \frac{n}{\lceil \frac{k}{2} \rceil} \rceil) \rceil. \end{cases}$$

Proof.

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Proof.

Lemma 2.13

$\beta_{[\pm 1]}(n, k) \in \Omega(\sqrt{\frac{k(n-k)}{n}})$.

Bisecting k-uniform families...

Theorem 2.14

Let c be a constant such that $0 < c < \frac{1}{2}$ and $n \in \mathbb{N}$. If $k \equiv 2 \pmod 4$ is odd, and $cn < k < (1-c)n$, then

$$\beta_{[\pm 1]}(n, k) \geq \delta n$$

, where $\delta = \delta(c)$ is some real positive constant. ▸ Proof.

Theorem 2.15

Let $\mathcal{A} = \binom{[n]}{k} \cup \binom{[n]}{k+1} \ldots \cup \binom{[n]}{n}$. Then,
$\frac{n-k+1}{2} \leq \beta_{[\pm 1]}(\mathcal{A}) \leq \min\{\frac{n}{2}, n-k+1\}$.

D-secting with one-sided error

$\beta_i(\mathcal{A})$: $\beta_D(\mathcal{A})$, when $D = \{i\}$.

Theorem 2.16

$\frac{n-i+1}{2} \leq \beta_i(n) \leq n - i + 1$, $n \in \mathbb{N}$, $i \in [n]$.

Moreover, $\beta_1(n) = \lceil \frac{n}{2} \rceil$.

Chapter 4
Induced bisecting families for hypergraphs

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- Further, if each $\forall B \in \mathcal{B}$, the number of colored points is $0 < d \leq n$ and $\langle X_A, Y_B \rangle = 0$:

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Riehl and Graham Evans Jr. (2003) Given the n quadratics in n variables $x_1(x_1 - 1), \ldots, x_n(x_n - 1)$ *with* 2^n *common zeros, the maximum number of those common zeros a polynomial P of degree k can go through without going through them all is* $2^n - 2^{n-k}$.

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Lemma 3.2

$\beta^d(n) \geq n-1$, *when d is odd.*

Lemma 3.3

Let d be an integer greater than 1. Then, $d \leq \beta^d(d+1) \leq d+1$. Moreover, $\beta^d(d+1) = d+1$, when d is even.

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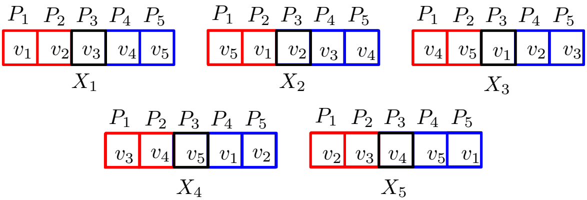

Figure : Vertices in (i) P_1 and P_2 are colored with $+1$, (ii) P_4 and P_5 are colored with -1; the vertex in P_3 remains uncolored. $\mathcal{Y} = \{Y_1, \ldots, Y_5\}$ is an induced bisecting family when $n = d+1 = 5$.

Theorem 3.4

Let $2 \leq d \leq n$, where d and n are integers. Then, $\frac{2n(n-1)}{d^2} \leq \beta^d(n) \leq \binom{\lceil \frac{2(n-1)}{d-1} \rceil}{2} + \lceil \frac{n-1}{d-1} \rceil (d+1)$. *Moreover,* $\beta^d(n) \geq n-1$, *when d is odd.*

Algorithm

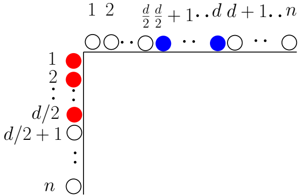

Figure : Coloring 1

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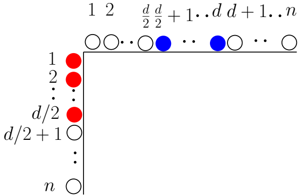

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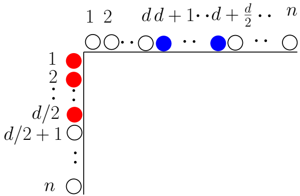
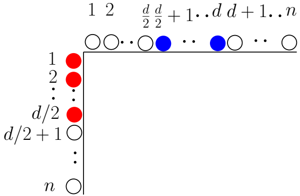


Figure : Coloring 2

Algorithm



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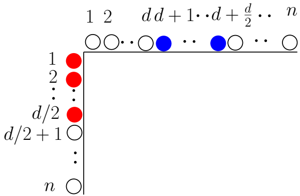

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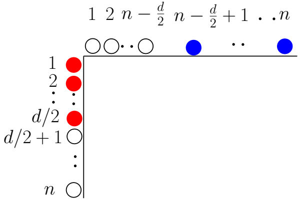
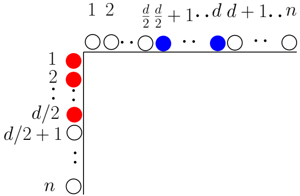


Figure : Coloring $\frac{2n}{d}-1$

Algorithm



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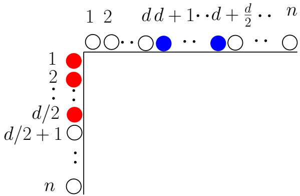

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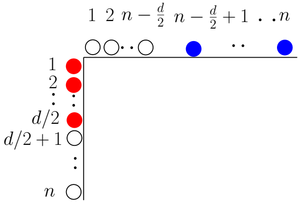

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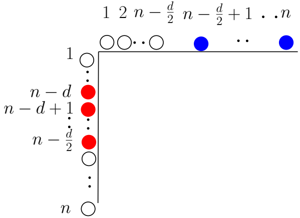

Figure : Coloring $\binom{\frac{2n}{d}}{2}$

Theorem 3.5

Let $2 \leq d \leq n$, where d and n are integers. Then,
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This establishes asymptotically tight bounds on $\beta^d(n)$ for all values of n, when d is odd. Moreover, the bound is asymptotically tight when $d \in O(\sqrt{n})$, even if d is even.
However, when $d \in \Omega(n^{0.5+\epsilon})$ and d is even, the above lower bound may not be asymptotically tight, for any ϵ, $0 < \epsilon \leq 0.5$.

Chapter 5
System of unbiased representatives for a set of bicolorings

Drug testing

n: a population of volunteers.

m: the number of attributes.

Drug testing

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m: the number of attributes.

Individual →

Attributes ↓

m \ n	1	2	3	4	5	6	7
Age>65?	-1	-1	1	1	1	1	1
Wt >55?	1	-1	1	-1	1	1	1
Ht $>5ft$?	1	1	1	1	-1	-1	-1
.							
.							

Figure : Person-Attribute matrix

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A subset chosen to represent an attribute must contain exactly equal number of individuals of complementary traits.

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$A = \{1, 2, 3, 4\}$ is an unbiased-representative for attributes age and weight, but not height.

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$Y_{height} = \{1, 1, 1, 1, -1, -1, -1\}$, $Y_{age} = \{-1, -1, 1, 1, 1, 1, 1\}$, $X_B = \{0, 1, 0, 1, 1, 1, 0\}$.

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$\langle Y_{height}, X_B \rangle = 0$, $\langle Y_{age}, X_B \rangle \neq 0$.

Definitions

Points: $\{1, \ldots, n\}$
Family of subsets: $\mathcal{A}$
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$$\gamma(n, [k_1, k_2], [r_1, r_2]) = \max_{\mathcal{B}} \gamma(\mathcal{B}, [k_1, k_2], [r_1, r_2]).$$

Results

Lemma 4.2

Let $F \in \mathbb{F}(x_1, \ldots, x_n)$ *be a polynomial and* $S_1, \ldots, S_n$ *be non-empty subsets of* $\mathbb{F}$*, for some field* $\mathbb{F}$*. If* F *vanishes on all but one point* $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n \subseteq \mathbb{F}^n$*, then deg(F)* $\geq \sum_{i=1}^{n}(|S_i| - 1)$*.*

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Theorem 4.3

$\gamma(n, [1, n-k], [2, n]) = n - 1$, *where* $1 \leq k \leq \lceil \frac{n}{2} \rceil$.

Results...

- $\gamma(\mathcal{B}, [1, n-1], [2, n])$: ▸ Hitting set connection.

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$$\frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}} \leq \gamma(n,k,r) \leq \frac{\binom{n}{k}}{\binom{r}{\frac{r}{2}}\binom{n-r}{k-\frac{r}{2}}}\left(1 + 0.7r + \ln(\binom{n-r}{k-\frac{r}{2}})\right).$$

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  provided $k \leq \log_4 \log_4(n^{0.5-\epsilon})$, for any $0 < \epsilon < 0.5$.

- $\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. Moreover, $\gamma(n, \frac{n}{2}, \frac{n}{2}) \geq \delta n$ if $n/2$ is even and $n/4$ is odd, for some $0 < \delta < 1$. Proof

Results...

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$$\frac{\binom{n}{k}}{\binom{2k}{k}} \leq \gamma(n,k,2k) \leq \frac{\binom{n}{k}}{\binom{2k}{k}}(1+o(1)),$$

provided $k \leq \log_4 \log_4(n^{0.5-\epsilon})$, for any $0 < \epsilon < 0.5$.

- $\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. Moreover, $\gamma(n, \frac{n}{2}, \frac{n}{2}) \geq \delta n$ if $n/2$ is even and $n/4$ is odd, for some $0 < \delta < 1$. ▸ Proof
- Hardness of approximation:

Results...

- The Rödl nibble extension Alon et al. (2003):

$$\frac{\binom{n}{k}}{\binom{2k}{k}} \leq \gamma(n, k, 2k) \leq \frac{\binom{n}{k}}{\binom{2k}{k}}(1 + o(1)),$$

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- $\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. Moreover, $\gamma(n, \frac{n}{2}, \frac{n}{2}) \geq \delta n$ if $n/2$ is even and $n/4$ is odd, for some $0 < \delta < 1$. Proof
- Hardness of approximation: $(1 - \Omega(1))\frac{\ln m}{4r}$.

Chapter 6

Bisection related problems

Definition 5.1 (Bisection closed families)

A family $\mathcal{A}$ consisting of even subsets of $[n]$ is called **bisection closed** if for each $A, B \in \mathcal{A}$, either A bisects B or B bisects A (or both).

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A family $\mathcal{A}$ consisting of even subsets of $[n]$ is called **bisection closed** if for each $A, B \in \mathcal{A}$, either A bisects B or B bisects A (or both).

Let $\vartheta(n)$ ($\vartheta(n, k)$) denote the maximum cardinality of any (respectively, a k-uniform) **bisection closed** family on $[n]$.

$\vartheta(n, k) = n :$

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A quadratic upper bound for $\vartheta(n)$ is easy.
If $\mathcal{A} = \{A_1, \ldots, A_t\}$ such that $|A_i| \leq |A_j|$ implies A_j bisects A_i, then $|\mathcal{A}| \leq n + 1$.

Theorem 5.2

Let n be an integer more than 30. Then $\vartheta(n) \leq \frac{8n(\ln n)^2}{\ln \ln n}$.

Proof: Let $\mathcal{A}$ be a bisection closed family of subsets of $[n]$ of maximum cardinality.

Theorem 5.2

Let n be an integer more than 30. Then $\vartheta(n) \leq \frac{8n(\ln n)^2}{\ln \ln n}$.

Proof: Let $\mathcal{A}$ be a bisection closed family of subsets of $[n]$ of maximum cardinality.

1. $\mathcal{A}_0 = \{A \in \mathcal{A} | |A| = 0 \mod 3\}$.
2. $\mathcal{A}_1 = \{A \in \mathcal{A} | |A| = 1 \mod 3\}$.
3. $\mathcal{A}_2 = \{A \in \mathcal{A} | |A| = 2 \mod 3\}$.

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Claim 1

$|\mathcal{A}_i| \leq n + 1$ *for* $i \in \{1, 2\}$.

Proof of $|\mathcal{A}_1| \leq n + 1$

Let $\mathcal{A}_1 = \{A_1, \ldots, A_m\}$ and let $a_1, \ldots, a_m$ denote their corresponding 0–1 incidence vectors.

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Construct m polynomials, f_1 to f_m, in the following way.

$$f_j(x) = \langle a_j, x \rangle - \frac{i}{2}, \text{ for } 1 \leq j \leq m,$$

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$$f_j(x) = \langle a_j, x \rangle - \frac{i}{2}, \text{ for } 1 \leq j \leq m,$$

Note that since $|A_i| \equiv i \pmod 3$, (i) $\langle a_i, a_i \rangle \equiv i \pmod 3$, (ii) $i \not\equiv \frac{i}{2} \pmod 3$ unless $i \equiv 0 \pmod 3$. So, f_j's are linearly independent over $\mathbb{F}_3$ (see (Jukna, 2011, Lemma 13.11)). This concludes the proof of the claim.

Size of $\mathcal{A}_0 = \{A \in \mathcal{A} | |A| = 0 \mod 3\}$ is still unknown.

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If we repeat the steps done above for each $p \in P$, then we can take care of sets of each cardinality in $\mathcal{A}$.
$|\mathcal{A}| \leq r \cdot p_r \cdot n$. □

Theorem 5.3

Let n be an even integer. Let $\mathcal{A}$ be a bisection closed family of maximum cardinality, where each $A \in \mathcal{A}$ has cardinality strictly greater than $\frac{n}{2}$ and $|A|$ is even. Then $|\mathcal{A}| \leq \frac{n}{2} + 1$.

Lemma 5.4 (Folklore)

Let $X_1, \ldots, X_m$ be unit vectors in $\mathbb{R}^n$ such that $\langle X_i, X_j \rangle \leq -\gamma$, for some $0 < \gamma < 1$ and $i \neq j$. Then, $m \leq \frac{1}{\gamma} + 1$.

Lemma 5.5

Let $Y_1, Y_2 \in \{-1, 1\}^n$ be incidence vectors corresponding to sets $A_1, A_2 \subseteq [n]$, where $Y(i) = 1$ if $i \in A$ and $Y(i) = -1$ otherwise. If A_1 bisects A_2, then $\langle Y_1, Y_2 \rangle = n - 2|A_1|$.

Proof.

If A_1 bisects A_2, then

$$\begin{aligned}
\langle Y_1, Y_2 \rangle =& (n - |A_1| - \frac{|A_2|}{2}) \cdot 1 && \text{(both } Y_1(i), Y_2(i) \text{ are -1)} \\
&+ \frac{|A_2|}{2} \cdot 1 && \text{(both } Y_1(i), Y_2(i) \text{ are 1)} \\
&+ (|A_1| - \frac{|A_2|}{2}) \cdot (-1) && (Y_1(i) \text{ is 1, } Y_2(i) \text{ is -1)} \\
&+ (\frac{|A_2|}{2}) \cdot (-1) && (Y_1(i) \text{ is -1, } Y_2(i) \text{ is 1)} \\
\implies \langle Y_1, Y_2 \rangle =& n - 2|A_1|.
\end{aligned}$$

□

Proof of Theorem 5.3

For any $A \in \mathcal{A}$, let $Y_A \in \mathbb{R}^n$ be defined as

$$Y_A(i) = \begin{cases} \frac{1}{\sqrt{n}}, & \text{if } i \in A \\ -\frac{1}{\sqrt{n}}, & \text{if } i \notin A. \end{cases} \tag{1}$$

In particular, $\|Y_A\|^2 = 1$. So, Y_A is a unit vector corresponding to A.

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In particular, $\|Y_A\|^2 = 1$. So, Y_A is a unit vector corresponding to A. From Lemma 5.5, we have the following observation regarding the dot products of distinct Y_A and Y_B.

$$\langle Y_A, Y_B \rangle = \begin{cases} \frac{n-2|A|}{n}, & \text{if } A \text{ bisects } B, \\ \frac{n-2|B|}{n}, & \text{if } B \text{ bisects } A. \end{cases} \tag{2}$$

Since $|A| > \frac{n}{2}$ and $|B| > \frac{n}{2}$, it follows that $\langle Y_A, Y_B \rangle \leq -\frac{2}{n}$. So, using Lemma 5.4, we get, $|\mathcal{A}| \leq \frac{n}{2} + 1$. □

Theorem 5.6

Let n be an even integer and let $\delta > 1$. Let $\mathcal{A}$ be a bisection closed family of maximum cardinality, where each $A \in \mathcal{A}$ has cardinality in the range $[\frac{n}{2} - \frac{\sqrt{n}}{2\delta}, \frac{n}{2} + \frac{\sqrt{n}}{2\delta}]$. and $|A|$ is even. Then, $|\mathcal{A}| \leq \frac{\delta^2}{\delta^2-1} n$.

Lemma 5.7

Alon (2009); Codenotti et al. (2000) Let A be an $m \times m$ real symmetric matrix with $a_{i,i} = 1$ and $|a_{i,j}| \leq \epsilon$ for all $i \neq j$. Let $tr(A)$ denote the trace of A, i.e., the sum of the diagonal entries of A. Let $rk(A)$ denote the rank of A. Then,

$$rk(A) \geq \frac{(tr(A))^2}{tr(A^2)} \geq \frac{m^2}{m + m(m-1)\epsilon^2}$$

.

Proof of Theorem 5.6

Let B be the $m \times n$ matrix with $Y_{A_1}, \ldots, Y_{A_m}$ as its rows, where

$$Y_A(i) = \begin{cases} \frac{1}{\sqrt{n}}, \text{ if } i \in A \\ -\frac{1}{\sqrt{n}}, \text{ if } i \notin A. \end{cases} \tag{3}$$

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Using Equation 2,
BB^T is an $m \times m$ real symmetric matrix with the diagonal entries being 1 and the absolute value of any other entry being $|\frac{n-2|A|}{n}| \leq \frac{1}{\delta\sqrt{n}}$.

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From Lemma 5.7, $rk(BB^T) \geq \frac{m}{1+\frac{m-1}{\delta^2 n}} > \frac{m}{1+\frac{m}{\delta^2 n}}$.

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We know that $rk(BB^T) \leq n$.

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We know that $rk(BB^T) \leq n$.
So, $n \geq m - \frac{m}{\delta^2}$ □

REFERENCES I

Martin Aigner, Günter M Ziegler, and Alfio Quarteroni. *Proofs from the Book*, volume 274. Springer, 2010.

N. Alon and Z. Füredi. Covering the cube by affine hyperplanes. *European Journal of Combinatorics*, 14(2):79–83, 1993. doi: 10.1006/eujc.1993.1011. URL https://www.scopus.com/inward/record.uri?eid=2-s2.0-38249006949&partnerID=40&md5=c606f9cd3dcf800edff600fc584f97f9.

Noga Alon. Combinatorial nullstellensatz. *Combinatorics, Probability and Computing*, 8(1-2):729, 1999. URL https://www.cambridge.org/core/article/combinatorial-nullstellensatz/E32C7ED059D6B92C6C67E18CC716872A.

Noga Alon. Perturbed identity matrices have high rank: Proof and applications. *Combinatorics, Probability and Computing*, 18(1-2):3–15, 2009.

Noga Alon, Béla Bollobás, Jeong Han Kim, and Van H. Vu. Economical covers with geometric applications. *Proceedings of the London Mathematical Society*, 86(2):273, 2003. doi: 10.1112/S0024611502013886. URL +http://dx.doi.org/10.1112/S0024611502013886.

László Babai and Péter Frankl. *Linear Algebra Methods in Combinatorics: With Applications to Geometry and Computer Science*. Department of Computer Science, univ. of Chicag, 1992.

Nikhil Bansal. Constructive algorithms for discrepancy minimization. In *51th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2010, October 23-26, 2010, Las Vegas, Nevada, USA*, pages 3–10, 2010. doi: 10.1109/FOCS.2010.7. URL http://dx.doi.org/10.1109/FOCS.2010.7.

Nikhil Bansal and Joel Spencer. Deterministic discrepancy minimization. *Algorithmica*, 67(4):451–471, 2013. doi: 10.1007/s00453-012-9728-1. URL http://dx.doi.org/10.1007/s00453-012-9728-1.

Manu Basavaraju, Mathew C Francis, M. S. Ramanujan, and Saket Saurabh. Partially polynomial kernels for set cover and test cover. *SIAM Journal on Discrete Mathematics*, 30(3):1401–1423, 2016.

József Beck and Vera T. Sós. Handbook of combinatorics (vol. 2). chapter Discrepancy Theory, pages 1405–1446. MIT Press, Cambridge, MA, USA, 1995. ISBN 0-262-07171-1. URL http://dl.acm.org/citation.cfm?id=233228.233235.

Béla Bollobás. *Extremal graph theory*. Dover Publications, INC, Mineola, New York, 2004.

Bernard Chazelle. *The Discrepancy Method*. Cambridge University Press, 2000. ISBN 9780511626371. URL http://dx.doi.org/10.1017/CBO9780511626371. Cambridge Books Online.

REFERENCES II

Bruno Codenotti, Pavel Pudlk, and Giovanni Resta. Some structural properties of low-rank matrices related to computational complexity. *Theoretical Computer Science*, 235(1):89 – 107, 2000. ISSN 0304-3975. doi: http://dx.doi.org/10.1016/S0304-3975(99)00185-1. URL http://www.sciencedirect.com/science/article/pii/S0304397599001851.

Robert Crowston, Gregory Gutin, Mark Jones, Saket Saurabh, and Anders Yeo. Parameterized study of the test cover problem. In *International Symposium on Mathematical Foundations of Computer Science*, pages 283–295. Springer, 2012.

Koen M. J. De Bontridder, B. J. Lageweg, Jan K. Lenstra, James B. Orlin, and Leen Stougie. Branch-and-bound algorithms for the test cover problem. In *European Symposium on Algorithms*, pages 223–233. Springer, 2002.

Pierre A. Devijver and Josef Kittler. *Pattern recognition: A statistical approach*, volume 761. Prentice-Hall London, 1982.

T. J. Dickson. On a problem concerning separating systems of a finite set. *J. Combin. Theory*, 7:191–196, 1969.

Irit Dinur and David Steurer. Analytical approach to parallel repetition. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 624–633. ACM, 2014.

Ronald Aylmer Fisher. An examination of the different possible solutions of a problem in incomplete blocks. *Annals of Human Genetics*, 10(1):52–75, 1940.

Michael R Garey and David S Johnson. Computers and intractability: a guide to the theory of np-completeness. 1979. *San Francisco, LA: Freeman*, 58, 1979.

Nils Hebbinghaus and Anand Srivastav. Multicolor discrepancy of arithmetic structures. In *A Panorama of Discrepancy Theory*, pages 319–424. Springer, 2014.

Rebecca Hoberg and Thomas Rothvoss. A logarithmic additive integrality gap for bin packing. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2616–2625. SIAM, 2017.

Stasys Jukna. *Extremal combinatorics: with applications in computer science*. Springer Science & Business Media, 2011.

G. Katona. On separating systems of a finite set. *J. Combin. Theory I*, pages 174–194, 1966.

Peter Keevash and Eoin Long. Frankl-rödl-type theorems for codes and permutations. *Transactions of the American Mathematical Society*, 369(2):1147–1162, 2017.

REFERENCES III

János Komlós, János Pach, and Gerhard Woeginger. Almost tight bounds for ϵ-nets. *Discrete & Computational Geometry*, 7(2):163–173, 1992. ISSN 1432-0444. doi: 10.1007/BF02187833. URL http://dx.doi.org/10.1007/BF02187833.

S. P. Lapage, Shoshana Bascomb, W. R. Willcox, and M. A. Curtis. Identification of bacteria by computer: general aspects and perspectives. *Microbiology*, 77(2):273–290, 1973.

Nathan Linial and Jaikumar Radhakrishnan. Essential covers of the cube by hyperplanes. *Journal of Combinatorial Theory, Series A*, 109(2):331 – 338, 2005. ISSN 0097-3165. doi: http://dx.doi.org/10.1016/j.jcta.2004.07.012. URL http://www.sciencedirect.com/science/article/pii/S0097316504001311.

L. Lovász. On the ratio of optimal integral and fractional covers. *Discrete Mathematics*, 13(4):383 – 390, 1975. ISSN 0012-365X. doi: http://dx.doi.org/10.1016/0012-365X(75)90058-8. URL http://www.sciencedirect.com/science/article/pii/0012365X75900588.

Shachar Lovett and Raghu Meka. Constructive discrepancy minimization by walking on the edges. In *Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on*, pages 61–67. IEEE, 2012.

Jiří Matoušek. Combinatorial discrepancy. In *Geometric Discrepancy*, volume 18 of *Algorithms and Combinatorics*, pages 101–135. Springer Berlin Heidelberg, 1999. ISBN 978-3-642-03941-6. doi: 10.1007/978-3-642-03942-3_4. URL http://dx.doi.org/10.1007/978-3-642-03942-3_4.

Bernard M. E. Moret and Henry D. Shapiro. On minimizing a set of tests. *SIAM Journal on Scientific and Statistical Computing*, 6(4):983–1003, 1985.

John E Olson and Joel H Spencer. Balancing families of sets. *Journal of Combinatorial Theory, Series A*, 25(1): 29–37, 1978.

R. W. Payne and D. A. Preece. Identification keys and diagnostic tables: A review. *Journal of the Royal Statistical Society. Series A (General)*, pages 253–292, 1980.

A. Rényi. On random generating elements of a finite boolean algebra. *Acta Sci. Math. (Szeged)*, 22(1-2):75–81, 1961.

REFERENCES IV

Emily Riehl and E. Graham Evans Jr. On the intersections of polynomials and the CayleyBacharach theorem. *Journal of Pure and Applied Algebra*, 183(13):293 – 298, 2003. ISSN 0022-4049. doi: http://dx.doi.org/10.1016/S0022-4049(03)00032-X. URL http://www.sciencedirect.com/science/article/pii/S002240490300032X.

Thomas Rothvoß. Approximating bin packing within o (log opt* log log opt) bins. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 20–29. IEEE, 2013.

Michael E. Saks. Slicing the hypercube. In *Surveys in Combinatorics, 1993*, pages 211–256. Cambridge University Press, 1993. ISBN 9780511662089. URL http://dx.doi.org/10.1017/CBO9780511662089.009.

David Saxton. Essential positive covers of the cube. *Journal of Combinatorial Theory, Series A*, 120(5):971 – 975, 2013. ISSN 0097-3165. doi: http://dx.doi.org/10.1016/j.jcta.2013.02.002. URL http://www.sciencedirect.com/science/article/pii/S0097316513000277.

J. Spencer. Minimal completely separating systems. *J. Combin. Theory*, 8:446–447, 1970.

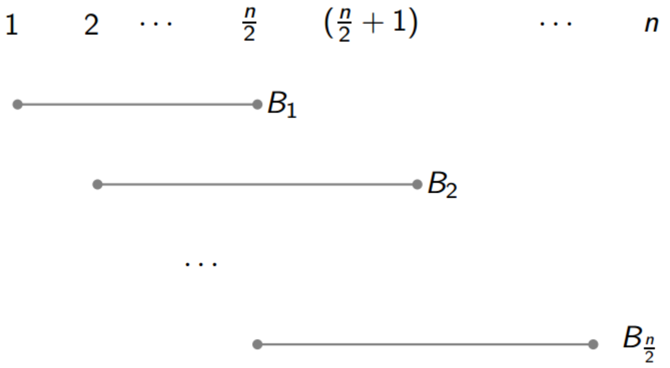
Joel Spencer. Six standard deviations suffice. *Trans. Amer. Math. Soc.*, 289(2):679–706, 1985. ISSN 0002-9947. doi: 10.2307/2000258. URL http://dx.doi.org/10.2307/2000258.

S. K. Stein. Two combinatorial covering theorems. *Journal of Combinatorial Theory, Series A*, 16(3):391 – 397, 1974. ISSN 0097-3165. doi: http://dx.doi.org/10.1016/0097-3165(74)90062-4. URL http://www.sciencedirect.com/science/article/pii/0097316574900624.

I. Wegener. On separating systems whose elements are sets of at most k elements. *Discrete Math.*, 28:219–222, 1979.

W. R. Willcox and S. P. Lapage. Automatic construction of diagnostic tables. *The Computer Journal*, 15(3): 263–267, 1972.

Proof of $\beta_{[\pm i]}(n) = \lceil \frac{n}{2i} \rceil$



Observe that $\mathcal{F}' = \{B_1, \ldots, B_{\frac{n}{2}}\}$ forms a bisecting family for $\mathcal{F} = 2^{[n]}$.

Upper bound for $\beta_{[\pm 1]}(n)$ (contd...)

Lemma 7.1

$\beta_{[\pm 1]}(n) \leq \frac{n}{2}$.

Upper bound for $\beta_{[\pm 1]}(n)$ (contd...)

Lemma 7.1

$\beta_{[\pm 1]}(n) \leq \frac{n}{2}$.

What about a lower bound for $\beta_{[\pm 1]}(n)$?

Upper bound for $\beta_{[\pm 1]}(n)$ (contd...)

Lemma 7.1

$\beta_{[\pm 1]}(n) \leq \frac{n}{2}$.

What about a lower bound for $\beta_{[\pm 1]}(n)$?

$\log(n)$,

Upper bound for $\beta_{[\pm 1]}(n)$ (contd...)

Lemma 7.1

$\beta_{[\pm 1]}(n) \leq \frac{n}{2}$.

What about a lower bound for $\beta_{[\pm 1]}(n)$?

$\log(n)$, $\Omega(\sqrt{n})$.

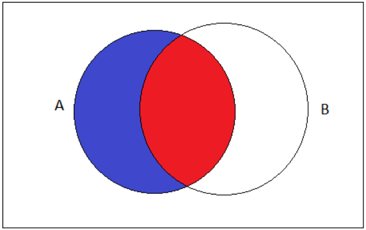
Lower bound for $\beta_{[\pm 1]}(n)$

Notations:
$X_A = (x_1, \ldots, x_n) \in \{0,1\}^n$: 0-1 incidence vector of A.
$R_A = (r_1, \ldots, r_n) \in \{-1,1\}^n$: (-1)-$(+1)$ incidence vector of A.

Lower bound for $\beta_{[\pm 1]}(n)$

Notations:

$X_A = (x_1, \dots, x_n) \in \{0,1\}^n$: 0-1 incidence vector of A.
$R_A = (r_1, \dots, r_n) \in \{-1,1\}^n$: (-1)-$(+1)$ incidence vector of A.



Observe that $\langle X_A, R_B \rangle$ is equivalent to $|A \cap B| - |A \cap ([n] \setminus B)|$.

For any even subset A_e, $\langle X_{A_e}, R_B \rangle \in \{0, \pm 2, \pm 4, \dots\}$ and for any odd subset A_o, $\langle X_{A_o}, R_B \rangle \in \{\pm 1, \pm 3, \pm 5, \dots\}$.

Lower bound for $\beta_{[\pm 1]}(n)$

Consider the polynomial M on $X = (x_1, \ldots, x_n) \in \{0,1\}^n$ as

$$M(X) = \prod_{B \in \mathcal{F}'} (\langle X, R_B \rangle)^2 - 1 \tag{4}$$

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$M(X)$ is (i) zero when $X = X_{A_o}$ for all odd subsets $A_o \in \mathcal{F}$; and (ii) positive when $X = X_{A_e}$ for all even subsets $A_e \in \mathcal{F}$.

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$deg(M'(X)) \leq deg(M(X)) = 2|\mathcal{F}'|$.

Definition 7.2

A multilinear polynomial $P(x_1, \ldots, x_n)$ *weakly represents* f if P is nonzero and for every $X = (x_1, \ldots, x_n)$ where $P(X)$ is nonzero, $sign(f(X)) = sign(P(X))$.

Definition 7.3

The *weak degree* of a function f is the degree of the lowest degree polynomial which weakly represents f.

Lemma 7.4 (Minsky, Papert, 1969 †)

The weak degree of the parity function on n variables is n.

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Note that $M'(X)$ *weakly* represents the *parity* function. This gives us, $n \leq deg(M'(X)) \leq deg(M(X)) = 2|\mathcal{F}'|$

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Tight bound for $\beta_{\pm i}(n)$

Lemma 7.5

$\beta_{\pm 1}(n) \geq \lceil \frac{n}{2} \rceil$.

Lemmas 7.1 and 7.5 imply

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Generalizing...

Theorem 7.7

$\beta_{\pm i}(n) = \lceil \frac{n}{2i} \rceil$.

Back

Proof of $\beta_{[\pm i]}(\mathcal{A}) \leq \frac{1}{2}\log m$ for $i \geq \sqrt{\frac{3n\ln(2m)}{t}}$

pick a set $\mathcal{B}$ of t random subsets $\{B_1, \ldots, B_t\}$ of $[n]$, where for each j, $1 \leq j \leq t$, a point $a \in [n]$ is chosen independently and uniformly at random into B_j.

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For any subset $A \in \mathcal{A}$, $|A \cap B_j| - |A \cap \overline{B_j}|$ can be viewed as sum of $|A|$ random variables distributed uniformly over $\{-1, 1\}$. Back

Proof of $\beta_{[\pm i]}(n,k) \geq \log(n-k+2)$ for $k \equiv 2 \pmod 4$

Let $\mathcal{B} = \{B_1, \ldots, B_t\}$ be a bisecting family for the family $\mathcal{A} = \binom{[n]}{k}$.

Proof of $\beta_{[\pm i]}(n,k) \geq \log(n-k+2)$ for $k \equiv 2 \pmod 4$

Let $\mathcal{B} = \{B_1, \ldots, B_t\}$ be a bisecting family for the family $\mathcal{A} = \binom{[n]}{k}$.
We associate a graph $G(\mathcal{F})$ in the following way:

$$V(G(\mathcal{F})) = \{S \in \binom{[n]}{\frac{k}{2}} : S \subseteq A,\ A \in \mathcal{F}\}$$

$$E(G(\mathcal{F})) = \{\{S_1, S_2\} : S_1 \cap S_2 = \emptyset, S_1, S_2 \in V(G(\mathcal{F}))\}.$$

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Observe that $G(\binom{[n]}{k})$ is the Kneser graph $KG(n, \frac{k}{2})$ having chromatic number $n-k+2$(see Bollobás (2004); Aigner et al. (2010)).

Proof of $\beta_{[\pm i]}(n, k) \geq \log(n - k + 2)$ for $k \equiv 2 \pmod 4$...

Contribution of each $B \in \mathcal{B}$ is a bipartite graph.

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Contribution of each $B \in \mathcal{B}$ is a bipartite graph.
Known result: The number of bipartite graphs needed to cover any graph is log of the chromatic number of the graph. Back

Proof of $\beta_{[\pm 1]}(n, k) \geq \delta n$

Let $\mathcal{C} \subseteq \{0, 1\}^n$ be a set of n-bit **binary** numbers together called a **binary code**.

$d(\mathcal{C})$: the **set of all allowed pairwise Hamming distances** in $\mathcal{C}$.

The code $\mathcal{C}$ is d-**avoiding** if $d \notin d(\mathcal{C})$.

Theorem 10.1 (Keevash, Long, 2017[†])

Let $\mathcal{C} \subseteq \{0, 1\}^n$ and let ϵ satisfy $0 < \epsilon < \frac{1}{2}$. Suppose that $\epsilon n < d < (1 - \epsilon)n$ and d is even. If $\mathcal{C}$ is d-avoiding, then $|\mathcal{C}| \leq 2^{(1-\delta)n}$, for some positive constant $\delta = \delta(\epsilon)$.

[†] Peter Keevash and Eoin Long. Frankl-rödl-type theorems for codes and permutations. *Transactions of the American Mathematical Society*, 369 (2): 1147–1162, 2017.

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Let $\mathcal{F} = \binom{[n]}{k}$ and let $\mathcal{F}' = \{B_1, B_2, \ldots\}$ be a bisecting family for $\mathcal{F}$ of the minimum cardinality.

For every $A \in \mathcal{F}$, there exists a $B \in \mathcal{F}'$ s.t. $|A \cap B| = \frac{k}{2}$ (an odd num).

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Let X_A, X_B denote the 0-1 n-dim incidence vector of A, B, resp.. Then, $< X_A, X_B > \equiv 1 \pmod 2$ when B bisects A (**since** $\frac{k}{2}$ **is odd**).

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Let W denote the vector space generated by the 0-1 incidence vectors of the sets in $\mathcal{F}'$ over $\mathbb{F}_2$. Let $W^{\perp}$ be the subspace which contains all the vectors perpendicular to W.

Proof of $\beta_{[\pm 1]}(n, k) \geq \delta n$ (contd...)

Observation: $W^{\perp}$ contains no vector of weight k.

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Reason: Suppose $X_A \in W^{\perp}$ has weight k. Then, from the definition of W, $\exists X_B \in W$, s.t. $< X_A, X_B > \equiv 1 \pmod 2$. This contradicts the definition of $W^{\perp}$.

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That means, for any $X_B, X_C \in W^{\perp}$, $X_B + X_C$ has weight $|B \triangle C| \neq k$.

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So, $dim(W^{\perp}) \leq n - \lfloor \delta n \rfloor$. It follows that $dim(W) \geq \lfloor \delta n \rfloor$.

Back.

Hitting set relation

Lemma 11.1

Let $\mathcal{B} = \{B_0, \ldots, B_{m-1}\} \subseteq \{-1, +1\}^n$ *be a family of bicolorings of* $[n]$*. Construct the family* $\mathcal{C} = \{C_1, \ldots, C_{2m}\}$ *where* $C_{2i+1} = B_i(+1)$ *and* $C_{2i+2} = B_i(-1)$*, for* $0 \leq i \leq m-1$*. Let* $H = \{h_1, h_2, h_3, \ldots\}$ *denote a hitting set for* $\mathcal{C}$*. Define* $\mathcal{A} = \{(h_1, h_q) | h_q \in H, q > 1\}$*. Then,* $\mathcal{A}$ *is a SUR for* $\mathcal{B}$ *of cardinality* $|H| - 1$*.*

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This also helps in establishing the inapproximability result for SUR using another result of (Dinur and Steurer, 2014).

Back.

Theorem 11.2

$\gamma(n, \frac{n}{2}, \frac{n}{2}) \leq \frac{n}{2}$. *Moreover,* $\gamma(n, \frac{n}{2}, \frac{n}{2}) \geq \delta n$ *if* $n/2$ *is even and* $n/4$ *is odd, for some* $0 < \delta < 1$.

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Using (Keevash and Long, 2017), the lower bound follows. □

Back.