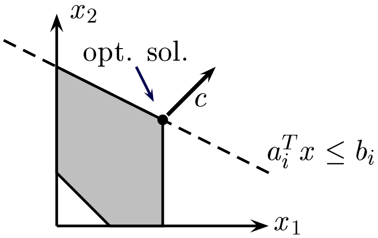
PART 8
INSERTION: LINEAR PROGRAMMING

SOURCE: *Geometric Algorithms and Combinatorial Optimization* (Grötschel, Lovász, Schrijver)

Linear programs

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ then

$$\begin{aligned} \max c^T x & & \\ Ax & \leq & b \\ x_i & \geq & 0 \ \forall i \end{aligned}$$



is called a linear program. Alternatively one might have

- min instead of max
- no non-negativity $x_i \geq 0$
- $Ax = b$

More terminology

- $\mathrm{conv}(\{x, y\}) := \{\lambda x + (1-\lambda) y \mid \lambda \in [0,1]\}$
- Set $Q \subseteq \mathbb{R}^n$ convex if $\forall x, y \in Q : \ \mathrm{conv}(\{x, y\}) \subseteq Q$
- A set P is called a polyhedron if $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
- If P bounded ($\exists M : P \subseteq [-M, M]^n$) then P is a polytope.

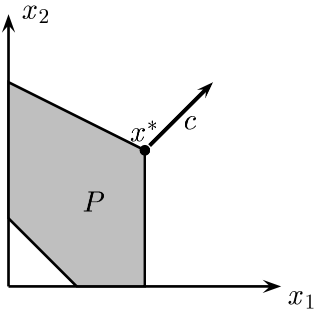
Vertices

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron.

Definition

A point $x^* \in P$ is called a vertex if there is a $c \in \mathbb{R}^n$ such that x^* is the unique optimum solution of $\max\{c^T x \mid x \in P\}$.

Alternative names: basic solution, extreme point.

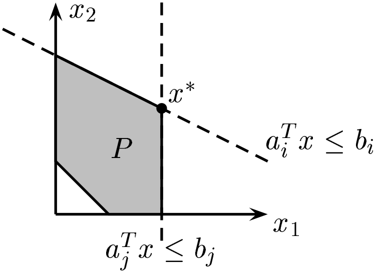


Alternative characterisations

Lemma

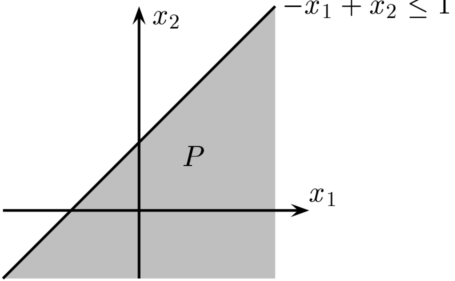
Let $x^* \in P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. The following statements are equivalent

- *x^* is a vertex*
- *There are no $y, z \in P$ with (x^*, y, z pairwise different) and $x^* \in conv\{y, z\}$*
- *There is a linear independent subsystem $A'x \leq b'$ (with n constraints) of $Ax \leq b$ s.t. $\{x^*\} = \{x \in \mathbb{R}^n \mid A'x = b'\}$.*



Not every polyhedron has vertices

Example: The polyhedron $P = \{x \in \mathbb{R}^2 \mid -x_1 + x_2 \leq 1\}$ does not have any vertices.



Lemma

Any polytope has vertices.

Lemma

Any polyhedron $P \subseteq \mathbb{R}^n$ with non-negativity constraints $x_i \geq 0\ \forall i = 1, \ldots, n$ has vertices.

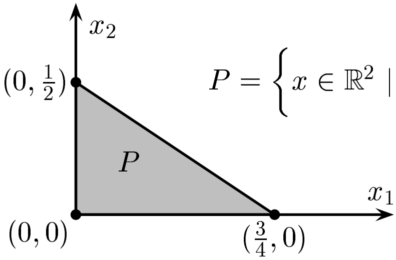
Support of vertex solutions

Lemma

Let x^* be a vertex of

$$P = \{x \in \mathbb{R}^n \mid a_j^T x \leq b_j \ \forall j = 1, \ldots, m; x_i \geq 0 \ \forall i\}$$

Then $|\{i \mid x_i^* > 0\}| \leq m$ (#non-zero entries $\leq$ #constraints).



$$P = \left\{ x \in \mathbb{R}^2 \mid \begin{array}{l} 4x_1 + 6x_2 \leq 3 \\ x_1 \geq 0; x_2 \geq 0 \end{array} \right\}$$

Proof: There is a subsystem I, J with $|J| + |I| = n$ and $\{x^*\} = \{x \mid a_j^T x = b_j \ \forall j \in J; \ x_i = 0 \ \forall i \in I\}$. Hence $|I| = n - |J| \geq n - m$.

Linear programming is doable in polytime

Theorem

Given $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$, *there is an algorithm which solves*

$$\max\{c^T x \mid Ax \leq b\}$$

in time polynomial in n, m *and the encoding length of* A, b, c. *The algorithm returns an optimum vertex solution if there is any.*

- Polynomial here means that the number of bit operations is bounded by a polynomial (Turing model).
- Encoding length (= #bits used to encode an object) for
  - integer $\alpha \in \mathbb{Z}$: $\langle \alpha \rangle := \lceil \log_2(|\alpha| + 1) \rceil + 1$.
  - rational number $\alpha = \frac{p}{q} \in \mathbb{Q}$: $\langle \alpha \rangle := \langle p \rangle + \langle q \rangle$
  - vector $c \in \mathbb{Q}^n$: $\langle c \rangle := \sum_{i=1}^{n} \langle c_i \rangle$
  - inequality $a^T x \leq \delta$: $\langle a \rangle + \langle \delta \rangle$
  - matrix $A = (a_{ij}) \in \mathbb{Q}^{m \times n}$: $\langle A \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ij} \rangle$

The ellipsoid method

Input: Fulldimensional polytope $P \subseteq \mathbb{R}^n$

Output: Point in P

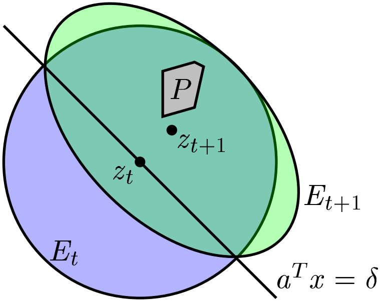
(1) Find ellipsoid $E_1 \supseteq P$ with center z_1

(2) FOR $t = 1, ..., \infty$ DO

(3) IF $z_t \in P$ THEN RETURN z_t

(4) Find hyperplane $a^T x = \delta$ through z_t such that $P \subseteq \{x \mid a^T x < \delta\}$

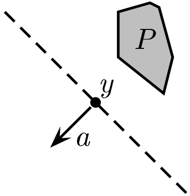
(5) Compute ellipsoid $E_{t+1} \supseteq E_t \cap \{x \mid a^T x \leq \delta\}$ with $\mathrm{vol}(E_{t+1}) = (1 - \frac{\Theta(1)}{n})\mathrm{vol}(E_t)$



The ellipsoid method (2)

Problem: SEPARATION PROBLEM FOR P:

- <u>Given:</u> $y \in \mathbb{Q}^n$
- <u>Find:</u> $a \in \mathbb{Q}^n$ with $a^T y > a^T x \; \forall x \in P$ (or assert $y \in P$).



Rule of thumb

If one can solve the SEPARATION PROBLEM for $P \subseteq \mathbb{R}^n$ in poly-time, then one can solve $\max\{c^T x \mid x \in P\}$ efficiently.

Important: The number of inequalities does <u>not</u> play a role. Especially we can optimize in many cases even if the number of inequalities is exponential.

Theorem

Let $P \subseteq \mathbb{R}^n$ be a polyhedron that can be described as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$, and let $c \in \mathbb{Q}^n$ be an objective function. Let φ be an upper bound on

- *the encoding length of each single inequality in $Ax \leq b$.*
- *the dimension n*
- *the encoding length of c.*

Suppose one can solve the following problem in time poly(φ):

> **Separation problem:** *Given $y \in \mathbb{Q}^n$ with encoding length poly(φ) as input. Decide, whether $y \in P$. If not find an $a \in \mathbb{Q}^n$ with $a^T y > a^T x \; \forall x \in P$.*

Then there is an algorithm that yields in time poly(φ) either

- *$x^* \in \mathbb{Q}^n$ attaining $\max\{c^T x \mid x \in P\}$ (x^* will be a vertex if P has vertices)*
- *P empty*
- *Vectors $x, y \in \mathbb{Q}^n$ with $x + \lambda y \in P \; \forall \lambda \geq 0$ and $c^T y \geq 1$.*

Here running times are w.r.t. the Turing machine model.

Weak duality

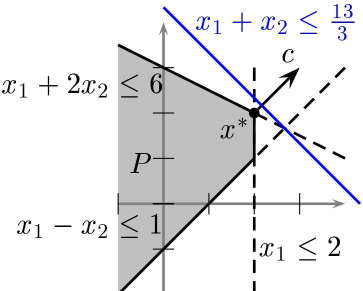
Observation

Consider the LP $\max\{c^T x \mid x \in P\}$ with $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Let $y \geq \mathbf{0}$. Then $(y^T A)x \leq y^T b$ is a feasible inequality for P (i.e. $(y^T A)x \leq y^T b \; \forall x \in P$). In fact, if $y^T A = c^T$, then

$$c^T x = (y^T A)x \leq y^T b \quad \forall x \in P$$

Example: $\max\{x_1 + x_2 \mid x_1 + 2x_2 \leq 6,\ x_1 \leq 2,\ x_1 - x_2 \leq 1\}$
Optimum solution: $x^* = (2, 2)$ with $c^T x^* = 4$.

$$\begin{array}{rcccl}
\frac{2}{3} \cdot (& x_1 & +2x_2 & \leq & 6) \\
0 \cdot (& x_1 & & \leq & 2) \\
\frac{1}{3} \cdot (& x_1 & -x_2 & \leq & 1) \\
\hline
 & x_1 & +x_2 & \leq & \frac{13}{3} \approx 4.33
\end{array}$$



Weak duality (2)

Theorem (Weak duality)

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. *Then*

$$\underbrace{\max\{c^T x \mid Ax \leq b\}}_{(P)} \leq \underbrace{\min\{b^T y \mid y^T A = c^T;\ y \geq \mathbf{0}\}}_{(D)}$$

given that both systems are feasible.

- If (P) is the primal program, then (D) is the dual program to (P).
- Note: The dual of the dual is the primal.

Strong duality

Theorem (Strong duality I)

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. Then

$$\max\{c^T x \mid Ax \leq b\} = \min\{b^T y \mid y^T A = c^T;\ y \geq \mathbf{0}\}$$

given that both systems are feasible.

Theorem (Strong duality II)

Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. Then

$$\max\{c^T x \mid Ax \leq b, x \geq \mathbf{0}\} = \min\{b^T y \mid y^T A \geq c^T, y \geq \mathbf{0}\}$$

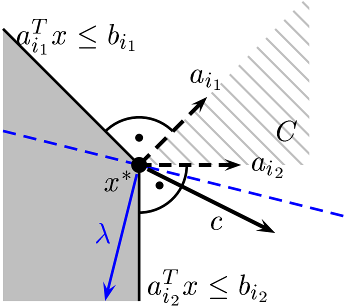
given that both systems are feasible.

Hand-waving proof of strong duality

Claim

Let x^* be optimum solution of $\max\{c^T x \mid Ax \leq b\}$. Then there is a $y \geq \mathbf{0}$ with $y^T A = c^T$ and $y^T b = c^T x^*$.

- Let $a_1, \ldots, a_m$ be rows of A.
- Let $I := \{i \mid a_i^T x^* = b_i\}$ be the tight inequalities.



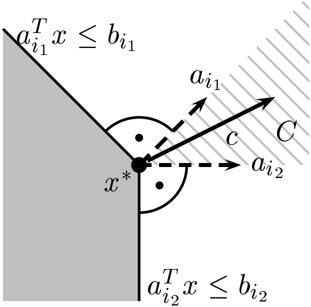
- Suppose for contradiction $c \notin \{\sum_i a_i y_i \mid y_i \geq 0, i \in I\} =: C$
- Then there is a $\lambda \in \mathbb{R}^n$ with $c^T \lambda > 0$, $a_i^T \lambda \leq 0 \; \forall i \in I$.
- Walking in direction λ improves objective function. But x^* was optimal. Contradiction!

Hand-waving proof of strong duality

Claim

Let x^* be optimum solution of $\max\{c^T x \mid Ax \leq b\}$. Then there is a $y \geq \mathbf{0}$ with $y^T A = c^T$ and $y^T b = c^T x^*$.

- Let $a_1, \ldots, a_m$ be rows of A.
- Let $I := \{i \mid a_i^T x^* = b_i\}$ be the tight inequalities.



- $\exists y \geq \mathbf{0} : y^T A = c^T$ and $y_i = 0 \; \forall i \notin I$ (we only use tight inequalities)

$$y^T b - c^T x^* = y^T b - y^T A x^* = y^T (b - A x^*) = \sum_{i=1}^{m} \underbrace{y_i}_{=0 \text{ if } i \notin I} \cdot \underbrace{(b_i - a_i^T x^*)}_{=0 \text{ if } i \in I} = 0$$

Complementary Slackness

Warning: Primal and dual are switched here.

Theorem (Complementary slackness)

Let x^* be a solution for

$$(P) : \min\{c^T x \mid Ax \geq b, x \geq \mathbf{0}\}$$

and y^* a solution for

$$(D) : \max\{b^T y \mid A^T y \leq c, y \geq \mathbf{0}\}.$$

Let a_i be the ith row of A and a^j be its jth column. Then x^* and y^* are both optimal $\Leftrightarrow$ both following conditions are true

- *Primal complementary slackness:* $x_j > 0 \Rightarrow (a^j)^T y = c_j$
- *Dual complementary slackness:* $y_i > 0 \Rightarrow a_i^T x = b_i$

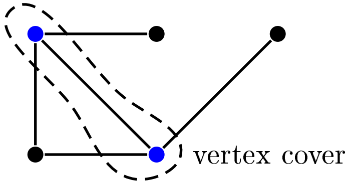
Part 9
Weighted Vertex Cover

Source: *Approximation Algorithms* (Vazirani, Springer Press)

Vertex Cover

Problem: Weighted Vertex Cover

- Given: Undirected graph $G = (V, E)$, node weights $c : V \to \mathbb{Q}_+$
- Find: Subset $U \subseteq V$ such that every edge is incident to at least one node in U and $\sum_{v \in U} c(v)$ is minimized.



Consider the LP

$$\min \sum_{v \in V} c(v) x_v$$

$$\begin{aligned} x_u + x_v &\geq 1 \quad \forall (u,v) \in E \\ x_v &\geq 0 \quad \forall v \in V \end{aligned}$$

Half-integrality

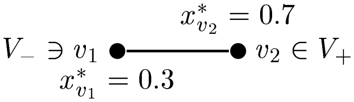
Lemma

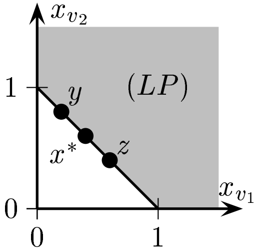
Let x^* be a basic solution of (LP). Then $x_v^* \in \{0, \frac{1}{2}, 1\}$ for all $v \in V$, i.e. x^* is half-integral.

- Suppose x^* is not half-integral, i.e. not both sets are empty:

$$V_+ := \left\{v \mid \frac{1}{2} < x_v^* < 1\right\}, V_- := \left\{v \mid 0 < x_v^* < \frac{1}{2}\right\}$$

- It suffices to show that x^* can be written as convex combination $x^* = \frac{1}{2}y + \frac{1}{2}z$ for 2 different feasible (LP) solutions y, z.

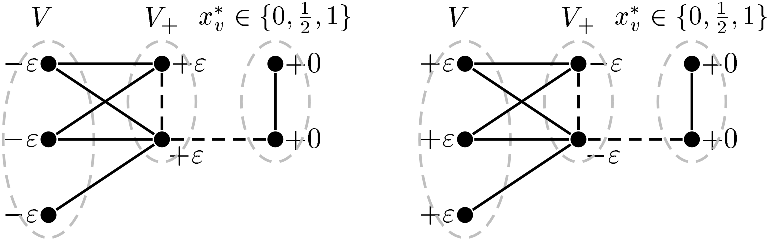




Half-integrality (2)

- Define

$$y_v := \begin{cases} x_v^* + \varepsilon & x_v^* \in V_+ \\ x_v^* - \varepsilon & x_v^* \in V_- \\ x_v^* & \text{otherwise} \end{cases} \quad \text{and} \quad z_v := \begin{cases} x_v^* - \varepsilon & x_v^* \in V_+ \\ x_v^* + \varepsilon & x_v^* \in V_- \\ x_v^* & \text{otherwise} \end{cases}$$



- Tight edges $(u, v) \in E : x_v^* + x_u^* = 1$ drawn solid
- Constraints satisfied by y, z for $\varepsilon > 0$ small enough. □

The Algorithm

Algorithm:

(1) Compute an optimum basic solution x^* to (LP)

(2) Choose vertex cover $U := \{v \mid x_v^* > 0\}$

Theorem

U is a vertex cover of cost $\leq 2 \cdot OPT_f$.

Proof.

Clearly U is feasible. Furthermore

$$\sum_{v \in U} c(v) = \sum_{v \in V} \lceil x_v^* \rceil c(v) \leq 2 \sum_{v \in V} x_v^* c(v) = 2 \cdot OPT_f.$$

□

Inapproximability

Theorem (Khot & Regev '03)

There is no polynomial time $(2-\varepsilon)$-apx unless Unique Games Conjecture is false.

Unique Games Conjecture

For all $\varepsilon > 0$, there is a prime $p := p(\varepsilon)$ such that the following problem is **NP**-hard:

- GIVEN: Equations $x_i \equiv_p a_{ij} x_j$ for some (i,j) pairs
- DISTINGUISH:
  - YES: max satisfiable fraction $\geq 1-\varepsilon$
  - NO: max satisfiable fraction $\leq \varepsilon$

Example:

$$
\begin{aligned}
x_1 &\equiv_{13} 4 \cdot x_3 \\
x_2 &\equiv_{13} 9 \cdot x_1 \\
&\cdots
\end{aligned}
$$

Part 7
Set Cover via LPs

Source: *Approximation Algorithms* (Vazirani, Springer Press)

A linear program for SetCover

Introduce decision variables

$$x_i = \begin{cases} 1 & \text{take set } S_i \\ 0 & \text{otherwise} \end{cases}$$

Formulate SetCover as integer linear program:

$$\begin{aligned} \min \sum_{i=1}^{m} c(S_i) x_i & & & \qquad (ILP) \\ \sum_{i: j \in S_i} x_i & \geq & 1 & \quad \forall j \in U \\ x_i & \in & \{0,1\} & \quad \forall i \end{aligned}$$

- Cheapest Set Cover solution = best (ILP) solution

The LP relaxation

We relax this to a linear program

$$\begin{aligned} \min \sum_{i=1}^{m} c(S_i) x_i & & & & (LP) \\ \sum_{i: j \in S_i} x_i &\geq & 1 & \quad \forall j \in U \\ 0 \leq x_i &\leq & 1 & \quad \forall i \end{aligned}$$

- (LP) can be solved in polynomial time (see next chapter)
- Let OPT_f be value of optimum solution
- Of course $OPT_f \leq OPT$
- Integrality gap

$$\alpha(n) := \sup_{\text{instances } |\mathcal{I}|=n} \frac{OPT(\mathcal{I})}{OPT_f(\mathcal{I})}$$

The algorithm

Algorithm:

(1) Solve $(LP) \rightarrow x^*$ opt. fractional solution

(2) (*Randomized rounding:*) FOR $i = 1, \ldots, m$ DO

  (3) Pick S_i with probability $\min\{\ln(n) \cdot x_i^*, 1\}$

(4) (*Repairing:*) FOR every not covered element $j \in U$ pick the cheapest set containing j

Analysis

Theorem

$E[APX] \leq (\ln(n) + 1) \cdot OPT_f$

Consider an element $j \in U$:

$$
\begin{aligned}
\Pr[j \text{ not covered in } (2)] &= \prod_{i: j \in S_i} \Pr[S_i \text{ not picked in } (2)] \\
&\leq \prod_{i: j \in S_i} (1 - \ln(n) \cdot x_i^*) \\
&\overset{1+y \leq e^y}{\leq} \prod_{i: j \in S_i} e^{-\ln(n) \cdot x_i^*} \\
&= e^{-\ln(n) \cdot \overbrace{\sum_{i: j \in S_i} x_i^*}^{\geq 1 \text{ due to LP ineq.}}} \\
&\leq e^{-\ln(n)} = \frac{1}{n}
\end{aligned}
$$

Analysis (2)

- Cost of randomized rounding:

$$E[\text{cost in (2)}] = \sum_{i=1}^{m} \Pr[S_i \text{ picked in (2)}] \cdot c(S_i)$$
$$\leq \sum_{i=1}^{m} \ln(n) x_i^* c(S_i) = \ln(n) \cdot OPT_f$$

- Cost of repairing step: In step (3), we pick n times with prob. $\leq \frac{1}{n}$ a set of cost $\leq OPT_f$. Hence

$$E[\text{cost of step (3)}] \leq n \cdot \frac{1}{n} \cdot OPT_f = OPT_f$$

- By linearity of expectation

$$E[APX] = E[\text{cost in (2)}] + E[\text{cost in (3)}] \leq (\ln(n)+1) \cdot OPT_f \quad \square$$